L

Power Series

C^∞ vs. analytic

Lecture 1
Mon, Jan 12
11am

A power series around $a \in \mathbb{C}$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

(always converges for $z=a$)

Ex 1) $\sum_{n=0}^{\infty} n!\, z^n$ converges only for $z=0$

$\sum_{n=0}^{\infty} n^n z^n$ —"—

2) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for every z

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$ —"—

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$

3) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$

converges for $|z| < 1$.

Recall:

Theorem For the power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

define $R \in [0, \infty]$ by

$$R = \frac{1}{\overline{\lim_n} |a_n|^{1/n}}$$

Then

a) If $|z-a| < R$, the series converges absolutely and uniformly for $|z-a| \le \rho < R$

b) If $|z-a| > R$, the terms are not bdd

c) If $r \in (0, R)$, then the series converges uniformly

$D_r(a) = \{ z \in \mathbb{C} : |z-a| < r \}$

R is called the radius of convergence.

Proof: exercise — use the root test.

Proposition Assume that $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have the radius of conv. $\geq r$. Then the power series $\sum_{n=0}^{\infty} c_n z^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, has a radius of conv. $\geq r$.

discrete convolution

Exercise: Idea of the proof:

Assume $|z| \leq r_0 < r$, where r_0 is fixed. Then

$$\sum_{n=0}^{\infty} |c_n| |z|^n \leq \left(\sum_{n=0}^{\infty} |a_n| r_0^n \right) \left(\sum_{n=0}^{\infty} |b_n| r_0^n \right).$$

ANALYTIC FUNCTIONS

Let $\Omega^{\text{open}} \subseteq \mathbb{C}$. A fn. $f: \Omega \to \mathbb{C}$ is (complex) differentiable at $z \in \Omega$ if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The value $f'(z)$ is called the (complex) derivative of f at z.

We say:

$$w = \lim_{h \to 0} f(h)$$

if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$|h| < \delta \ \& \ h \neq 0 \Rightarrow |w - f(h)| < \varepsilon$$

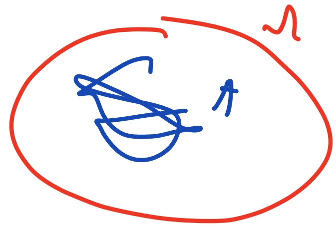
Let $\Omega \neq \emptyset$ be open in Ω.

Def A fn $f: \Omega \to \mathbb{C}$ is analytic (holomorphic) in Ω if it is differentiable in Ω (i.e., diff at every $z \in \Omega$).

Remark We don't assume the continuity of f'.
We will prove that f' is cont. and in fact analytic.

If $A \subseteq \mathbb{C}$, $A \neq \emptyset$, is not nec. open
then we say that f is analytic on A if
it is a restriction of an analytic fn defined
on some open set $\Omega \supseteq A$.

A typical example: $A = \{z_0\}$
(f is analytic at z_0 if it is analytic in some nbhd
of $\{z_0\}$)



Without rigorously stating: (exercise)
Sums, differences, products of analytic fn's are analytic.
Same for the quotient at the set where the denominator
is nonzero.

Composition: Ω: always nonempty, open

Proposition Assume that f, g are analytic in Ω, G
resp., assume $f(\Omega) \subseteq G$. Then $g \circ f$ is analytic
on Ω and $(g \circ f)'(z) = g'(f(z))\, f'(z)$.

Lecture 2
Mon, Jan 12
2-3pm

We would like to do

$$(g\circ f)'(z) = \frac{(g\circ f)(z+h)-(g\circ f)(z)}{f(z+h)-f(z)} \quad \frac{f(z+h)-f(z)}{h} \qquad (*)$$

Proof Let $z\in\Omega$. It is sufficient to prove that every sequence $\{h_n\}\to 0$, $h_n\neq 0$, there exists a subsequence h_{n_k} s.t.

$$\frac{1}{h_{n_k}}\left(g(f(z+h_{n_k})) - g(f(z))\right) \longrightarrow g'(f(z))\, f'(z) \qquad (**)$$

Case 1 $f(z)\neq f(z+h_n)$ for all except finitely many h_n.
Then we can use $(*)$.

Case 2 $f(z) = f(z+h_n)$ for ∞-many n.
passing to a subseq, we may assume that this holds for all n.
Then

$$\frac{g(f(z+h_n)) - g(f(z))}{h_n} = 0 \qquad \forall n$$

and

$$g'(f(z))\, f'(z) = 0.$$

So $(**)$ holds for this sequence. ∎

In the proof we used that f is continuous at z, which follows from differentiability.

Def A function f analytic in $\mathbb{C}$ is called *entire*.

Examples of analytic/entire functions:

Ex 1) $(z^n)' = n z^{n-1}$ (exercise)

$\therefore$ polynomials are entire

2) $e^z = e^{x+iy} \overset{\text{def}}{=} e^x(\cos y + i \sin y)$ where $z = x+iy$, $x, y \in \mathbb{R}$

Since $e^{z_1+z_2} = e^{z_1} e^{z_2}$ $\forall z_1, z_2 \in \mathbb{C}$, we only have to check

$$\lim_{z \to 0} \frac{e^z - 1}{z} = 1$$

Exercise: Prove this (this is a complex limit!)

Important alternative use the Cauchy-Riemann equation.

3) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ are entire. ※

e^z is periodic w. period $2\pi i$. ($2\pi i$ smallest period)
$(e^{2\pi i} = \cos(2\pi) + i \sin 2\pi = 1)$

$\sin z$, $\cos z$ period w. period 2π.

To discuss roots and $\log z$, we need an inverse function theorem.

complex IFT

Theorem Ω, G open subsets of $\mathbb{C}$. Assume that $f: \Omega \to \mathbb{C}$ is cont. at z, and let $g: G \to \mathbb{C}$ be s.t. $f(\Omega) \subseteq G$ & $g(f(z)) = z$ $\forall z \in \Omega$. If g is differentiable at $f(z) \in \mathbb{C}$ and $g'(f(z)) \neq 0$, then f is differentiable at z and $f'(z) = \dfrac{1}{g'(f(z))}$.

Proof Let $h \neq 0$ be small. Then

$$1 = \frac{g(f(z+h)) - g(f(z))}{h}$$

$$= \frac{g(f(z+h)) - g(f(z))}{f(z+h) - f(z)} \quad \frac{f(z+h) - f(z)}{h}$$

Note that $f(z+h) - f(z) \neq 0$ since f is 1-1.

$\left(g(f(z)) = z \right.$
$g(f(z+h)) = z+h$
$\left. \therefore f(z+h) \neq f(z) \right)$

Since

$$\lim_{h \to 0} \left(f(z+h) - f(z) \right) = 0$$

(continuity of f at z)

$$\lim \frac{g(f(z+h)) - g(f(z))}{f(z+h) - f(z)} = g'(f(z)) \neq 0$$

So $\displaystyle\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists and equals $\dfrac{1}{g'(f(z))}$. ☐

Logarithm

Problem: e^z is not bijective; moreover, it is periodic.
Thus it does not have a uniquely defined inverse.

Def Let $f : \Omega \to \mathbb{C}$, where $\Omega \overset{open}{\subseteq} \mathbb{C}$,
be cont. and such that

$$z = \exp(f(z)), \quad z \in \Omega.$$

Then f is called a branch of the logarithm.

Since

$$e^{z+2\pi i} = e^{z} \qquad \forall\, z \in \mathbb{C}$$

we have: If f, g are two branches of the log on $\Omega^{\text{open}} \subseteq \mathbb{C}$, then

$$f(z) = g(z) + 2\pi k i$$

where $k \in \mathbb{Z}$ is fixed.

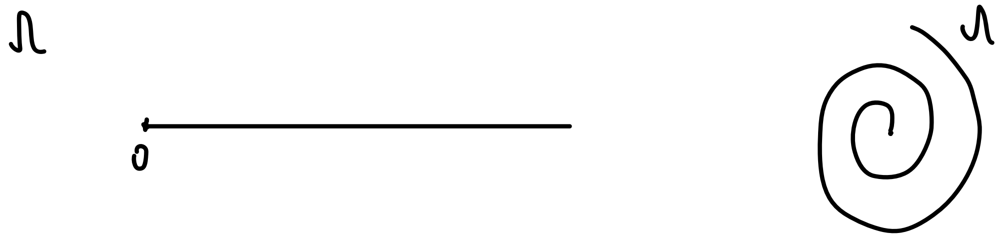
Conversely, if f on Ω is a branch of log, then so is $f(z) + 2\pi k i$, where $k \in \mathbb{Z}$.

This says: Fixing Ω basically fixes the branch, up to multiples of $2\pi i$.

Note: Different Ω's give different logs.

$\Omega = \mathbb{C} \setminus (-\infty, 0]$

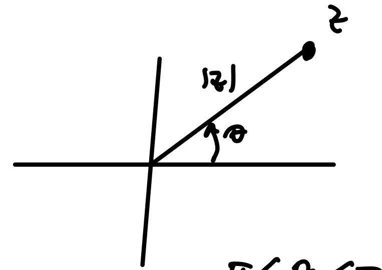
lead to different log's



Many times, we'll use the following branch of the log called the principal branch of the logarithm.

Let

$$\Omega = \mathbb{C} \setminus (-\infty, 0]$$



$-\pi < \theta < \pi$

Then represent $z \in \Omega$ as

$$z = |z| e^{i\theta} \qquad \text{(polar representation)}$$

where $-\pi < \theta < \pi$. Note: θ is a continuous fn. of $z \in \mathbb{C}$). Then let

$$f(re^{i\theta}) = \log r + i\theta \qquad r > 0, \; -\pi < \theta < \pi$$

This is called the principal branch. Indeed:

$$e^{f(re^{i\theta})} = e^{\log r + i\theta} = r e^{i\theta}$$

$\therefore$

$$\exp(f(z)) = z$$

Principal branch:

$$\log z = \log r + i\theta$$

where $\theta = \arg z$.

Important remark: To define the log, we need to define $\arg z$ for $z \in \Omega$

Lecture 3
Wed, Jan 14, 2pm

Prop. Let f be a branch of the log in Ω. Then f is analytic in Ω and $f'(z) = \frac{1}{z}$.

Follows from the complex IFT.

$i^i = ?$

Powers: For any fixed $a \in \mathbb{C}$.

$$z^a = \exp(a \log z)$$

Important: We cannot define log as a (single-valued) fn. in $\mathbb{C} \setminus \{0\}$.
(Hint: $\arg z$ would have to be cont.)

Fact: A fn. f of x, y can be considered a fn. g of $z, \bar{z}$ by writing $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$.
Such fn. is analytic in some Ω open iff $\frac{\partial}{\partial \bar{z}} g(z, \bar{z}) = 0$
(exercise) (direct from CR)

($\bar{\partial}$ equation f analytic $\Leftrightarrow$ $\bar{\partial} f = 0$.)

CAUCHY – RIEMANN EQUATIONS

Assume that

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. Let $u = \operatorname{Re} f$, $v = \operatorname{Im} f$ (standard notation : $f = u + iv$)

Also st. not. $z = x + iy$.

Consider $h \to 0$ along the real values. Then

$$\lim_{h\to 0,\, h\in\mathbb{R}} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0,\, h\in\mathbb{R}} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h\to 0,\, h\in\mathbb{R}} \frac{v(x+h,y) - v(x,y)}{h}$$

$$= \frac{\partial u}{\partial x}(x,y) + i\,\frac{\partial v}{\partial x}(x,y)$$

$$\overset{\text{def}}{=} u_x(x,y) + i\, v_x(x,y)$$

$\therefore$ u_x, v_x exist at z and

$$f' = u_x + i v_x$$

Now, consider $h \to 0$ along $i\mathbb{R}$. Changing h to ih:

$$f'(z) = \lim_{h\to 0,\, h\in\mathbb{R}} \frac{f(z+ih) - f(z)}{ih}$$

$$= \lim_{h\to 0,\, h\in\mathbb{R}} \frac{u(x,y+h) - u(x,y)}{ih} + i \lim_{h\to 0,\, h\in\mathbb{R}} \frac{v(x,y+h) - v(x,y)}{ih}$$

$$= v_y(x,y) - i\, u_y(x,y)$$

$\therefore$

$$f' = v_y - i u_y$$

Comparing the two formulas, we obtain the Cauchy–Riemann eq's

$$u_x = v_y \qquad (CR)$$
$$u_y = -v_x$$

how about if we relax this assumption

Theorem Let $u, v : \Omega \to \mathbb{R}$, where $\Omega^{\text{open}} \subseteq \mathbb{C}$, and let

$$f = u + iv : \Omega \to \mathbb{C}$$

If f is analytic in Ω, then u, v satisfy (CR) in Ω.

Conversely, assume that $u, v \in C^1(\Omega)$ satisfy the (CR). Then $f = u + iv$ is analytic in Ω.

Proof Necessity (of analy.) already proven.

Define

$$\varphi(h,k) = \underbrace{u(x+h, y+k) - u(x,y)}_{\text{this is } \operatorname{Re}(f(z+h+ik) - f(z))} - h\,u_x(x,y) - k\,u_y(x,y)$$

this expression goes to 0 as $o(|h|+|k|)$

$$= u(x+h, y+k) - u(x, y+k) - h\,u_x(x,y)$$
$$+ u(x, y+k) - u(x,y) - k\,u_y(x,y)$$

By the MVT, $\exists\, h_1 \in (\min\{h,0\}, \max\{h,0\})$,
$\exists\, k_1 \in (\min\{k,0\}, \max\{k,0\})$ s.t.

$$\varphi(h,k) = h\,u_x(x+h_1, y+k) - h\,u_x(x,y)$$
$$+ k\,u_y(x, y+k_1) - k\,u_y(x,y)$$

$$\Rightarrow \lim_{h+ik \to 0} \frac{\varphi(h,k)}{h+ik} = 0$$

We have used $\lim_{h+ik \to 0} \frac{h}{h+ik} = 0$.

So, we have proven

$$u(x+h, y+k) - u(x,y) = u_x(x,y)\, h + \overbrace{u_y(x,y)}^{-v_x(x,y)}\, k + \varphi(h,k)$$

and similarly

$$v(x+h, y+k) - v(x,y) = v_x(x,y)\, h + \underbrace{v_y(x,y)}_{u_x(x,y)}\, k + \psi(h,k)$$

with

$$\lim_{h+ik\to 0} \frac{\varphi(h,k)}{h+ik} = 0 \qquad \text{and} \qquad \lim_{h+ik\to 0} \frac{\psi(h,k)}{h+ik} = 0$$

$\therefore$

$$\lim_{h+ik\to 0} \frac{f(z+h+ik) - f(z)}{h+ik} = u_x(z) + i v_x(z) + \underbrace{\lim_{h+ik\to 0} \frac{\varphi(h,k) + i\psi(h,k)}{h+ik}}_{=0}$$

where we have used

$$u_x h - v_x k + i(v_x h + u_x k) = (h+ik)(u_x + i v_x). \qquad \blacksquare$$

Ex $\quad f(z) = e^z = u + iv$

where

$$u = e^x \cos y$$
$$v = e^x \sin y$$

Exercise: Check that the (CR) holds. In addition, $u, v \in C^1(\mathbb{R}^2)$. $\quad \therefore$ f entire. $\blacksquare$

Let $f = u + iv$ be analytic, and assume $u, v \in C^2(\Omega)$.

then

$$u_{xx} = (v_y)_x \overset{C^2}{=} (v_x)_y = (-u_y)_y = -u_{yy}$$

$\therefore$

$$u_{xx} + u_{yy} = 0$$

$\therefore$ u is harmonic

Similarly (exercise): v is harmonic

Or use

$f = u + iv$

$\rightarrow$ $-if = v - iu$

$\therefore$ v is harmonic

So: f is harmonic

Lecture 4
Wed, Jan 21

We have a connection:

elliptic PDEs in 2D $\leftrightarrow$ theory of analytic function

Def If u, v harmonic in Ω (open) and if $f = u + iv$ is analytic in Ω, then v is conjugate harmonic to u

Note: If v conj. harmonic to u, then $-u$ is conjugate harmonic to v $\quad (f = u + iv \Rightarrow -if = v - iu)$

Given a harmonic fn, does there exist a conjugate harmonic fn.

Important (counter) example:

Consider

$$u = \frac{1}{2}\log(x^2+y^2)$$

$(= \log|x+iy|$
Recall: $\log z = \log|z| + i\arg z$
if $\log$ exists$)$

This is a harmonic function

If the conjugate harmonic exists, it must be of the form

$$v = \arg(x+iy)$$

If $u+iv_1$ and $u+iv_2$ are analytic, then $i(v_1-v_2)$ is analytic. Denote $v=v_1-v_2$. Then

iv is analytic

By the C-R equations:

$$v_x=0$$
$$v_y=0$$

$\therefore$ $v\equiv$ const. We have proven:

Prop If v_1, v_2 are conjugate harmonic to same u, then v_1-v_2 is constant.

Thus, if $u=\frac{1}{2}\log(x^2+y^2)$ has a harm conjugate, it must be of the form $\arg(x+iy)$ (Why? – consider data)

Therefore u does not have a harmonic conj. in $\mathbb{C}\setminus\{0\}$.

Reason: Cannot define $\arg(x+iy)$ a cont. function.

Def Notation $\mathbb{D}_r(z_0)=B_r(z_0)\ldots$ disk around z_0 w/ radius $r>0$

$\mathbb{D}_r=\mathbb{D}_r(0)$, $\mathbb{D}=\mathbb{D}_1$.

We always assume $r>0$.

Theorem Let $\Omega=\mathbb{D}_r(z_0)$, where $r>0$, $z_0\in\mathbb{C}$.
Let u be harmonic in Ω. Then $\exists$ harmonic conjugate to u.

Exercise: If Ω is not simply connected (s.c = the complement has at least one bdd component)

then this is not true

Exercise: Consider actually the following proof for simply conn. domain.

Proof Plan: Suppose that v exists. We'll get a formula, which we'll then check it works.

WLOG, $\Omega = \mathbb{D}_r$. Since $v_y = u_x$, we have

$$v(x,y) = \int_0^y u_x(x,t)\,dt + v(x,0)$$

To determine $v(x,0)$, we differentiate in x:

$$v_x = \int_0^y u_{xx}(x,t)\,dt + v_x(x,0)$$

$\Rightarrow$ (CR)

$$-u_y(x,y) = -\int_0^y u_{yy}(x,t)\,dt + v_x(x,0)$$

$$= -u_y(x,y) + u_y(x,0) + v_x(x,0)$$

Therefore,

$$v_x(x,0) = -u_y(x,0)$$

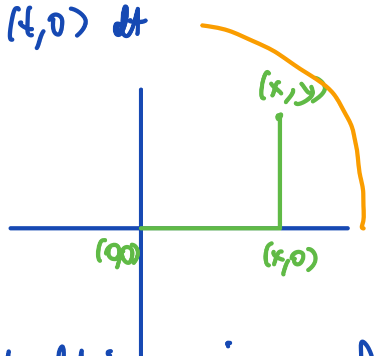
$\Rightarrow$

$$v(x,0) = -\int_0^x u_y(t,0) + v(0,0)$$

Set $v(0,0)=0$ (v is determined up to a constant)

$\therefore$

$$v(x,y) = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(t,0)\,dt$$



Note that both integrals are well-defined since $\Omega = \mathbb{D}$.

Check the CR equations

1) $\quad v_x(x,y) = \int_0^y u_{xx}(x,t)\,dt - u_y(x,0)$

$\quad = -\int_0^y u_{yy}(x,t)\,dt - u_y(x,0)$

$\quad = -u_y(x,y) + \cancel{u_y(x,0)} - \cancel{u_y(x,0)}$

$\quad = -u_y$

2) $\quad v_y = u_x$

Another application of the CR.

Prop. Let f be an analytic fn in Ω. If either

a) f' is constantly zero

b) f maps to a line $\quad$ ($f(\Omega)$ is a subset of a line)

c) f maps to a circle $\quad$ ($f(\Omega)$ —"— of a circle)

in Ω, then f is constant.

$u^2+v^2=r^2$

Differentiate in x & y

b, c : exercise

Proof Assume $f'=0$. Since $f' = u_x + i v_x$, we get

$u_x=0$ & $v_x=0$ in Ω. By the C-R eq's,

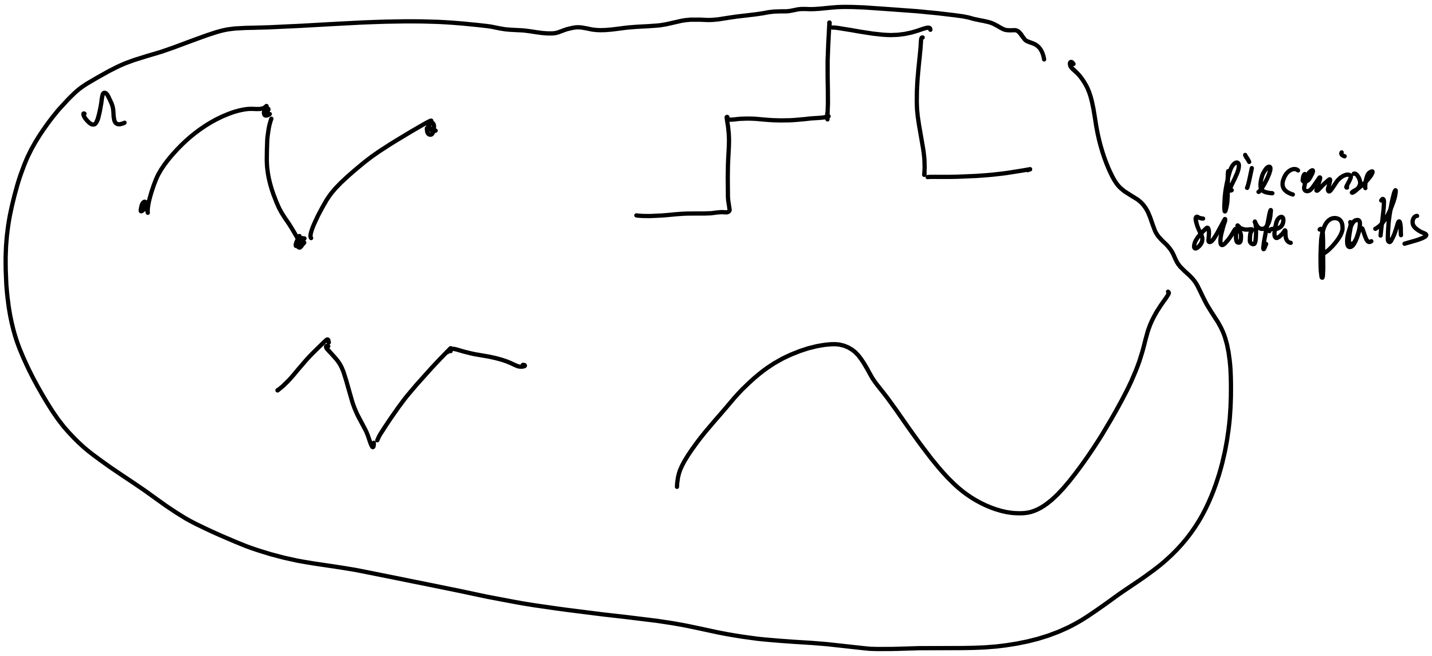
$u_x=0$ & $u_y=0$ $\quad \therefore u \equiv$ const.

Analytic Functions as Mappings

We'll prove that analytic f preserves angles at all points z_0 where $f'(z_0) \neq 0$.

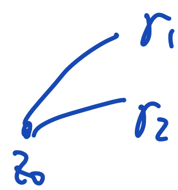
Lecture 5
Fri, Jan 23, 26
2pm

A path in $\Omega \subseteq \mathbb{C}$ is a continuous fn. $\gamma : [a,b] \to \Omega$ where $-\infty < a < b < \infty$. If γ' exists for every $t \in [a,b]$ and γ' is cont., we call γ a C^1 (or smooth) path. A path γ is piecewise C^1 (or piecewise smooth) if $\exists$ partition $a = t_0 < t_1 < \dots < t_n = b$ s.t. γ is C^1 on each $[t_{j-1}, t_j]$ for $j = 1, \dots, n$.



Def Let γ_1, γ_2 be two smooth curves s.t.

$$\gamma_1(t_1) = \gamma_2(t_1) = z_0$$



The angle between γ_1 and γ_2 at z_0 is defined as

$$\arg \gamma_2'(t_1) - \arg \gamma_1'(t_1) \in \mathbb{R} / 2\pi\mathbb{Z}$$

Now, assume that $\gamma : \Omega \to \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ analytic. Then

$$\tilde{\gamma} = f \circ \gamma$$

is smooth $(= C^1)$ and

$$\tilde{\gamma}'(t) = f'(\gamma(t))\, \gamma'(t) \qquad (*)$$

which is a complex chain rule. (Exercise: counterexample when f not analytic: $f(z) = \bar{z}$)

To prove this, we can follow the proof of the chain rule, or the calculus:

$$\tilde{\gamma} = (u \circ (\gamma_1, \gamma_2), v \circ (\gamma_1, \gamma_2)) \qquad (*) \text{ (identify } \gamma = \gamma_1 + i\gamma_2 = (\gamma_1, \gamma_2) \text{)}$$

Then

$$\tilde{\gamma}' = (u_x \gamma_1' + u_y \gamma_2', \; v_x \gamma_1' + v_y \gamma_2')$$

$$\overset{CR}{=} (u_x \gamma_1' - v_x \gamma_2', \; v_x \gamma_1' + u_x \gamma_2')$$

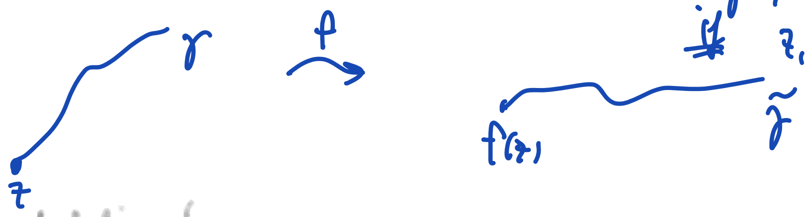
$$= (u_x + i v_x)(\gamma_1' + i\gamma_2')$$

From $(*)$, we get

$$\arg \tilde{\gamma}'(t) = \arg f'(\gamma(t)) + \arg \gamma'(t)$$

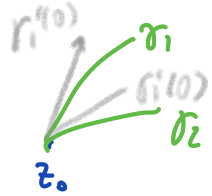
(Recall: $\arg(z_1 z_2) = \arg z_1 + \arg z_2$) if $z_1, z_2 \neq 0$

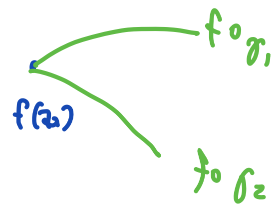
this says the angles are preserved if



we avoid arg 0 ← not defined!

Let γ_1, γ_2 be two curves s.t. $\gamma_1(0) = \gamma_2(0) = z_0$





Then, if $f'(z_0) \neq 0$ and $\gamma_1'(0), \gamma_2'(0) \neq 0$, which we can always assume, then

$$\arg \tilde{\gamma}_1'(f(z_0)) - \arg \tilde{\gamma}_2'(f(z_0)) = \arg \gamma_1'(0) - \arg \gamma_2'(0)$$

where
$$\tilde{\gamma}_1 = f \circ \gamma_1$$
$$\tilde{\gamma}_2 = f \circ \gamma_2 .$$

Ω always assumed open, nonempty

Thm Suppose $f : \Omega \to \mathbb{C}$ is analytic. Then f preserves angles at every point $z_0 \in \Omega$ s.t. $f'(z_0) \neq 0$.

signed angle

From the chain rule, we also get

$$|\tilde{\gamma}'(f(z_0))| = |f'(z_0)| \, |\gamma'(t_0)|$$

$\therefore$ An analytic fn. multiplies $|\gamma'(t_0)|$ with a fixed ft.

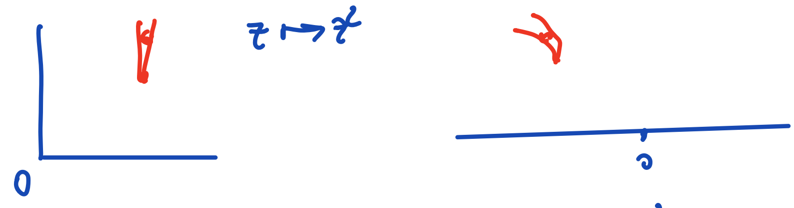
signed?

Def A fn. $f : \Omega \to \mathbb{C}$, which preserves angles in the sense (**) and s.t.

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

exists for every $z_0 \in \Omega$ is called conformal.

Ex



$z \mapsto z^2$ is conformal in $\{x,y>0\}$.

We have proven that analytic fn.'s are conformal at all pts where the derivative is nonzero.

Ex $f(z)=\bar{z}$ is not conformal even though it preserves the size of the angles.

Later: If $f'(z_0)=0$ and $f \not\equiv$ const, then f multiplies angles at that point by a positive integer, which is the multiplicity of the zero of $f(z)-f(z_0)$.

Let's consider the converse.

Assume that f is s.t. $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are cont. ($f \in C^1$), and assume that it preserves angles between curves.

Let

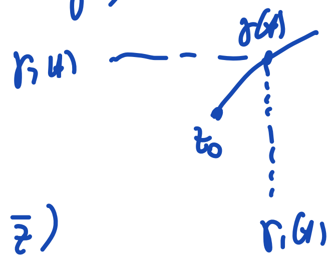
$$\tilde{\gamma}(t) = f(\gamma(t)) = f(\gamma_1, \gamma_2)$$

where

$$\gamma(t) = \underset{\in \mathbb{R}}{\gamma_1(t)} + i \underset{\in \mathbb{R}}{\gamma_2(t)}$$

Then by the real-variable chain rule

$$\tilde{\gamma}'(t) = f_x \underbrace{\gamma_1'(t)}_{\frac{1}{2}(\gamma' + \bar{\gamma}')} + f_y \underbrace{\gamma_2'(t)}_{\frac{1}{2i}(\gamma' - \bar{\gamma}')}$$



Let $z = \gamma'(0)$. Then

$$\tilde{\gamma}'(f(z_0)) = f_x \frac{1}{2}(z + \bar{z}) + f_y \frac{1}{2i}(z - \bar{z})$$

$\Rightarrow$

$$\tilde{\gamma}'(f(z_0)) = \frac{1}{2}(\overset{\text{at } z_0}{f_x} - i f_y) z + \frac{1}{2}(f_x + i f_y)\bar{z} \qquad (*)$$

Assume $\gamma'(0) \neq 0$. If angles are preserved, then

$$\arg \frac{\tilde{\gamma}'(f(z_0))}{\gamma'(0)} \quad \text{is independent of} \quad \arg \gamma'(0)$$

This fact and $(*)$ imply that

$$\frac{1}{2}(f_x - i f_y) + \frac{1}{2}(f_x + i f_y)\frac{\bar{z}}{z}$$

has a constant argument regardless of the choice of $z \in \mathbb{C} \setminus \{0\}$

Now we consider the situation:

$$a + bm$$

where $a, b \in \mathbb{C}$, has the same argument regardless of the choice of $m \in \partial \mathbb{D}$.

Note that $a + bm$ traces a circle of radius $|b|$ around a. Thus b must be zero.

Applying this fact,

$$f_x + i f_y = 0$$

(more precisely: $f_x(z_0) + i f_y(z_0) = 0$)

This is the complex form of the C-R equation

$$f = u + iv$$

or

$$(u+iv)_x + i(u+iv)_y = 0$$

$$\therefore \quad u_x - v_y + i(v_x + u_y) = 0$$

$$\therefore \quad \begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}$$

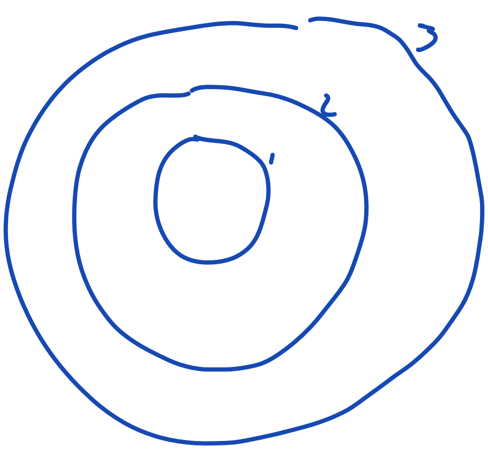
(We can also go back.) $\therefore$ f must be analytic by a thm from before

C-R equation $f_x + i f_y = 0$

LINEAR FRACTIONAL TRANSFORMATIONS

Lecture 6
Mon, Jan 26, 26
2pm

We work on $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ — Riemann sphere / Möbius plane



Looman-Menchoff theorem

1. C-R w/o C^1
2. Riemann sphere v Möbius plane

Standard: Möbius plane

L-M thm: If f is cont. & CR hold except on a countable set of pts, then f is holomorphic

Def A mapping

$$z \mapsto Sz = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ is called a <u>linear fractional transformation</u>

If $ad - bc \neq 0$, this is called a Möbius transformation.

$$ad - bc = 0 \iff S \text{ is constant}$$

We'll show first (or via a computation) that the inverse of Möbius is Möbius.

We can represent

$$z \mapsto Sz = \frac{az+b}{cz+d} \quad \text{as} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

note: the same S can be represented with several matrices (multiply by a nonzero constant)

Consider

$$S_1 z = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad S_2 z = \frac{a_2 z + b_2}{c_2 z + d_2}$$

then

$$S_1 S_2 z = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$$

$$\sim \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Exercise: Make a formal statement.

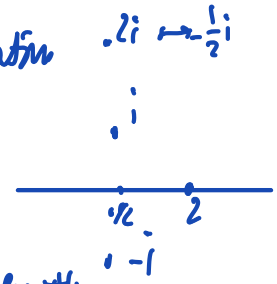
Composition corresponds to a product.

We see: The inverse of $z \mapsto \frac{az+b}{cz+d}$ exists iff $ad - bc \neq 0$, and the inverse is linear

Special ones:

$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \sim z \mapsto z+\alpha$ translation

$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \sim z \mapsto kz$ rotation & dilation



$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim z \mapsto \frac{1}{z}$ inversion

in the sense of complex #'s

$$\frac{a(z+\frac{d}{c})}{c(z+\frac{d}{c})}$$

If $a, c \neq 0$, we can write

$$\frac{az+b}{cz+d} = \frac{az + \frac{da}{c}}{cz+d} + \frac{b-\frac{d}{c}a}{cz+d}$$

$$= \frac{a}{c} + \frac{\frac{b}{c} - \frac{d}{c^2}a}{z+\frac{d}{c}}$$

This is a composition of translation, inversion, rotation, dilation, translation

If $a=0$, $c \neq 0$:

$$\frac{b}{cz+d} = \frac{\frac{b}{c}}{z+\frac{d}{c}}$$

translation, inversion, rotation, dilation

If $c=0$, $a \neq 0$,

$$\frac{az+b}{d} = \frac{a}{d}\left(z+\frac{b}{a}\right)$$

translation, rotation, dilation

Every Möbius is a composition of

translation, inversion, rotation, dilation in some order with repetitions

<u>Exercise</u>: Is it true that we can do it w/o repetition FOR FUN

CROSS RATIO

Möbius transformations have three complex degrees of freedom.

Prop Given $z_2, z_3, z_4 \in \mathbb{C}_\infty$, different,
$\exists!$ linear transformation S s.t.

$$S : \begin{aligned} z_2 &\longmapsto 1 \\ z_3 &\longmapsto 0 \\ z_4 &\longmapsto \infty \end{aligned}$$

$Sz = (z, z_2, z_3, z_4)$
cross ratio

Proof (existence)

cross ratio

$$Sz = \frac{\dfrac{z-z_3}{z-z_4}}{\dfrac{z_2-z_3}{z_2-z_4}} \overset{def}{=} (z, z_2, z_3, z_4)$$

if none are infinity

If $z_2 = \infty$: $\quad Sz = \dfrac{z-z_3}{z-z_4}$

If $z_3 = \infty$ $\quad Sz = \dfrac{z_2-z_4}{z-z_4}$

If $z_4 = \infty$ $\quad Sz = \dfrac{z-z_3}{z_2-z_3}$ $\blacksquare$

Using the existence, we can map different z_2, z_3, z_4 to different w_2, w_3, w_4 by composing S from the proposition with the inverse of

$Sz = (z, z_2, z_3, z_4)$

$$Tz = (z, w_2, w_3, w_4)$$

i.e.,

$$T^{-1}S$$

Proof (uniqueness)

Assume

$$T, S: \quad z_2 \mapsto 1, \quad z_3 \mapsto 0, \quad z_4 \mapsto \infty$$

Then "putting them on the same side"

$$ST^{-1}z : \quad 1 \mapsto 1, \quad 0 \mapsto 0, \quad \infty \mapsto \infty$$

Let

$$ST^{-1}z = \frac{az+b}{cz+d}$$

Then

$1 \mapsto 1$: $\frac{a+b}{c+d} = 1$ $\quad \therefore \quad a+b = c+d$

$0 \mapsto 0$ $\quad \frac{b}{d} = 0$ $\quad \therefore \quad b=0, \ d \neq 0$

$\infty \mapsto \infty$: $\quad a \neq 0, \ c=0$

$\therefore \quad a = d, \quad b = c = 0$

$$ST^{-1}z = \frac{az}{a} = z \qquad \forall z.$$

$\therefore$

$$Sz = Tz \qquad \forall z.$$

Proposition If z_1, z_2, z_3, z_4 distinct and T Möbius then

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$$

Proof Denote

$$Sz = (z, z_2, z_3, z_4) \qquad (*)$$

(maps $z_2 \mapsto 1$, $z_3 \mapsto 0$, $z_4 \mapsto \infty$). Then

$$ST^{-1}: \quad Tz_2 \mapsto 1$$
$$Tz_3 \mapsto 0$$
$$Tz_4 \mapsto \infty \quad \text{by the def'n of the cross ratio}$$

So

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (ST^{-1})(Tz_1) = Sz_1 \overset{*}{=} (z_1 z_2, z_3, z_4). \quad \blacksquare$$

Assigned HW #1.

Lecture 7
Wed, Jan 28, 26

Thm z_1, z_2, z_3, z_4 distinct. Then

$(z_1, z_2, z_3, z_4) \in \mathbb{R} \iff z_1, z_2, z_3, z_4$ belong to a circle

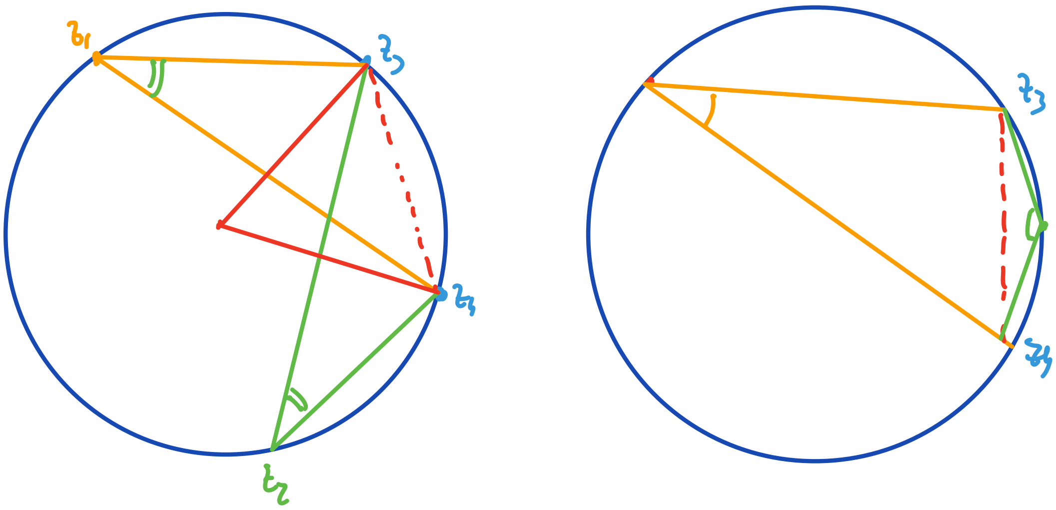
Recall: line is also a circle – it is a circle which passes through ∞.

Exercise: Alhfors says that then follows from $\frac{az+b}{cz+d}$ maps $\mathbb{R}$ to a subset of a circle.

"Geometric proof"

$$\arg (z_1, z_2, z_3, z_4) = \arg \frac{z_1 - z_3}{z_1 - z_4} - \arg \frac{z_2 - z_3}{z_2 - z_4}$$

If z_1, z_2, z_3, z_4 lie on a circle, then this difference is 0 or $+\pi$ $(\pm 2k\pi)$ based



Exercise
Work up the geometric proof.

better: This geometric part actually follows from the analytic proof.

Proof Let
$$Tz = \frac{az+b}{cz+d}$$
be Möbius ($ad - bc \neq 0$) Then we claim:

1) If $a\bar{c} - c\bar{a} = 0$, then
$$Tz \in \mathbb{R} \iff z \text{ belongs to the line } (a\bar{d} - \bar{b}c)z + (b\bar{c} - \bar{a}d)\bar{z} + b\bar{d} - \bar{b}d = 0$$

2) If $a\bar{c} - c\bar{a} \neq 0$, then
$$Tz \in \mathbb{R} \iff \left| z + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - a\bar{c}} \right| = \left| \frac{ad - bc}{\bar{a}c - \bar{c}a} \right|$$

Assume
$$Tz \in \mathbb{R}$$

Then
$$\frac{az+b}{cz+d} = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}$$

$\iff$
$$(a\bar{c} - \bar{a}c)z\bar{z} + (a\bar{d} - \bar{b}c)z + (b\bar{c} - \bar{a}d)\bar{z} + b\bar{d} - \bar{b}d = 0$$

If
$$a\bar{c} - c\bar{a} = 0$$
we get
$$(a\bar{d} - \bar{b}c)z + (b\bar{c} - \bar{a}d)\bar{z} + b\bar{d} - \bar{b}d = 0$$
This is either an empty set, or a point, or a line.
It must be a line since it is an infinite set (consider T)
If $a\bar{c} - \bar{a}c \neq 0$, then
$$|z|^2 + \frac{a\bar{d} - c\bar{b}}{a\bar{c} - \bar{a}c} z + \frac{b\bar{c} - \bar{a}d}{a\bar{c} - \bar{a}c}\bar{z} + \frac{b\bar{d} - \bar{b}d}{a\bar{c} - \bar{a}c} = 0$$

which we write as (skip)

$$\left(z+\frac{\bar{a}d-\bar{c}b}{\bar{a}c-a\bar{c}}\right)\left(\bar{z}+\frac{a\bar{d}-c\bar{b}}{a\bar{c}-c\bar{a}}\right)=\frac{\bar{b}d-b\bar{d}}{a\bar{c}-\bar{a}c}-\frac{(\bar{a}d-\bar{c}b)(a\bar{d}-c\bar{b})}{(\bar{a}c-a\bar{c})^2}$$

$$\overset{\text{skip}}{=}\left|\frac{ad-bc}{\bar{a}c-\bar{c}a}\right|^2$$

$\therefore$

$$\left|z+\frac{\bar{a}d-\bar{c}b}{\bar{a}c-a\bar{c}}\right|=\left|\frac{ad-bc}{\bar{a}c-\bar{c}a}\right|$$

which is what we claimed.

Note that: we can go back, too.

($\Rightarrow$) Let

$$Tz=(z,z_2,z_3,z_4)=\frac{az+b}{cz+d}$$

Assume $Tz_1\in\mathbb{R}$ (means $(z_1,z_2,z_3,z_4)\in\mathbb{R}$)

Then z_1 belongs to the circle from the claim

But note that z_2,z_3,z_4 also belong to the same circle.

That's because z_2,z_3,z_4 go to real #'s $(1,0,\infty)$

so they belong to this same circle.

($\Leftarrow$) Let T be as above. Assume z_1,z_2,z_3,z_4 belong to a circle. But we know that z_2,z_3,z_4 belong to the circle in the claim.

Therefore z_1 belongs to the circle from the claim

Going backwards in the proof of the claim Tz_1 must be real. Then use $Tz_1=(z_1,z_2,z_3,z_4)$. ∎

Corollary A Möbius transformation maps circles to circles.

Proof Apply the invariance of the cross ratio under a Möbius transformation. ∎

Question: If an ~~analytic~~ meromorphic fn map circles to circles, does it have to be Möbius.

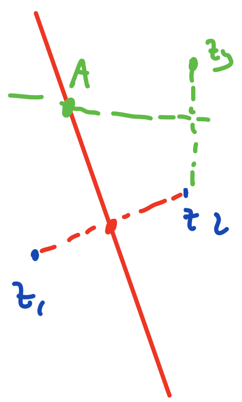
Will know how to do this later.

Def: z and z^* are *symmetric* w/r to the circle through (distinct) z_1, z_2, z_3 if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

As it stands now, the def'n depends on the choice of z_1, z_2, z_3 (we'll show that this is the case).

To prove that z_1, z_2, z_3 determine a circle, use the picture



(the center) = (intersection of two lines)

Note that the operation $z \leftrightarrow z^*$ is symmetric.

We need to check that the def'n does not depend on the choice of z_1, z_2, z_3.

The proof is by mapping the circle to the line then $z_1, z_2, z_3 \in \mathbb{R}$ (even pick), and the symmetry means

$$\frac{\dfrac{z^* - z_2}{z^* - z_3}}{\cancel{\dfrac{z_1 - z_2}{z_1 - z_3}}} = \frac{\dfrac{\bar{z} - z_2}{\bar{z} - z_3}}{\cancel{\dfrac{z_1 - z_2}{z_1 - z_3}}}$$

$$\Leftrightarrow \quad \frac{z^* - z_2}{z^* - z_3} = \frac{\bar{z} - z_2}{\bar{z} - z_3}$$

M is 1-1

$$\Leftrightarrow \quad z^* = \bar{z}$$

No dependence on z_1, z_2, z_3.

Lecture 8
Fri, Jan 30, 26

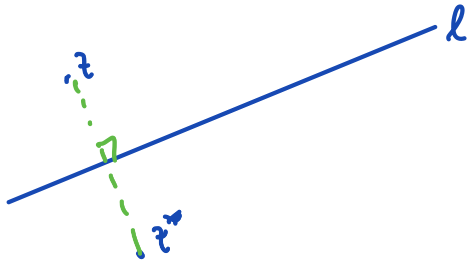
If we consider the circle $\mathbb{R}$ ($\cup \{\infty\}$), then the def'n implies $z^* = \bar{z}$.

Note the following: If z^* and z are symmetric w.r. to a circle C, and T is Möbius, then Tz^* and Tz are symmetric w.r. to the circle TC.

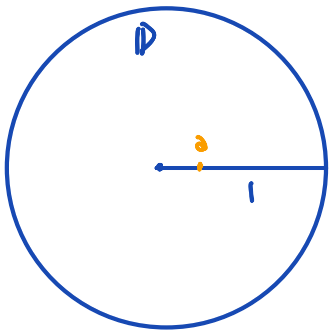
This is by!

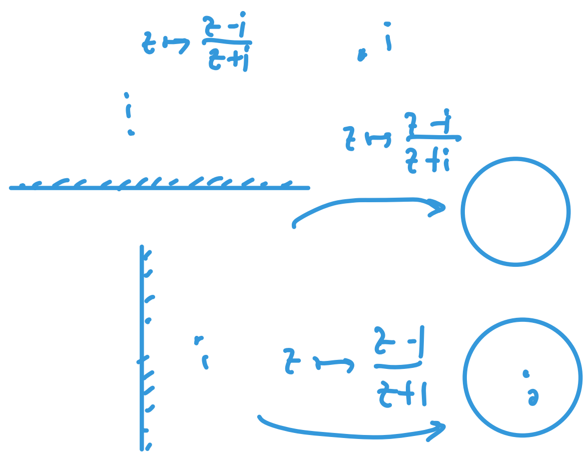
$$(Tz, Tz_1, Tz_2, Tz_3) = (z, z_1, z_2, z_3)$$

Using that translation and multiplication (or translations, dilations, and rotations), we get:



How about the symmetry w.r.t. a circle C?
By reduction, we only need to consider $\partial \mathbb{D}$ and $z = a \in (0,1)$.

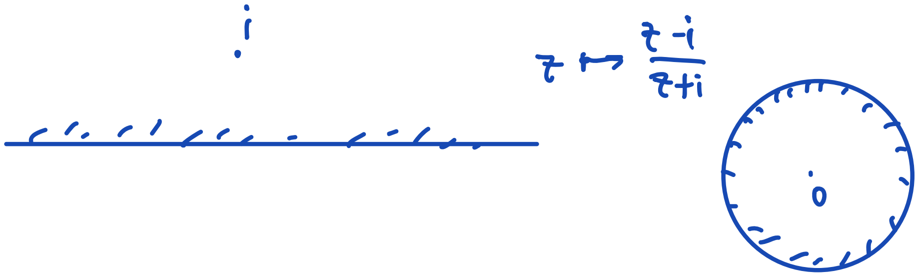




Then use

$$\mathbb{H} = \{z : \operatorname{Im} z > 0\}$$

and map

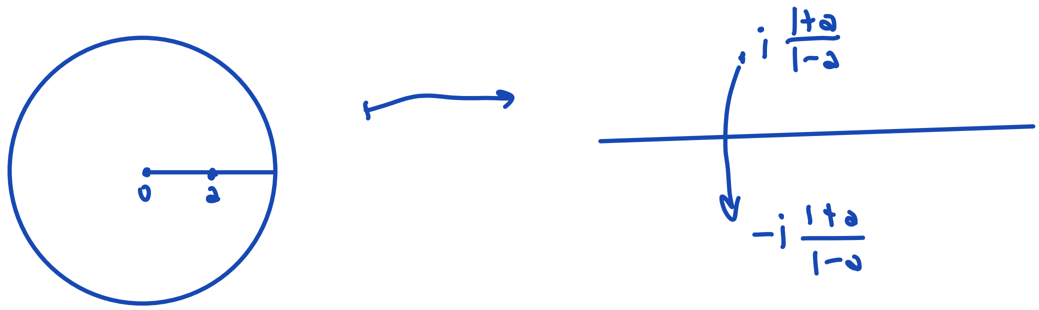


Inverse

$$S: z \mapsto i\,\frac{1+z}{1-z}$$

check: $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} =$

$= \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}$ ~ corresponds to

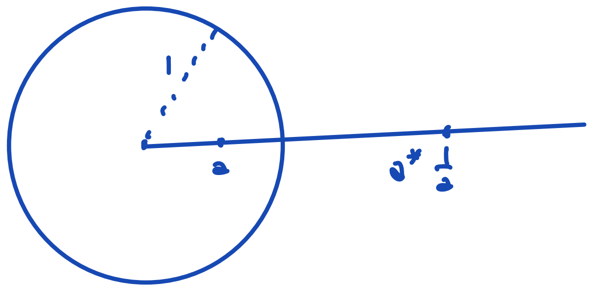


We need to map $-i\frac{1+a}{1-a}$ back:

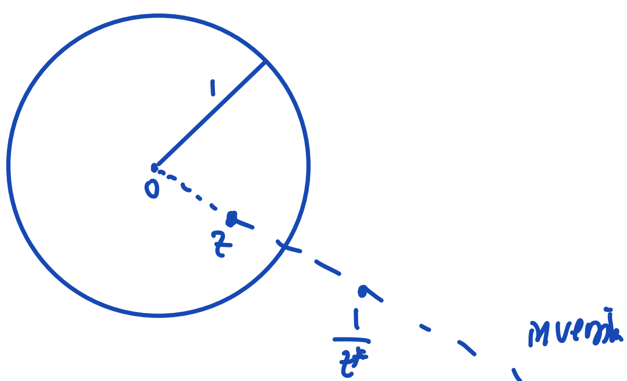
$$a^* = S^{-1}\left(-i\frac{1+a}{1-a}\right) = \frac{-i\frac{1+a}{1-a} - i}{-i\frac{1+a}{1-a} + i} = \frac{\frac{1+a}{1-a}+1}{\frac{1+a}{1-a}-1} = \frac{1+a+1-a}{1+a-1+a} = \frac{2}{2a} = \frac{1}{a}$$

$\therefore$

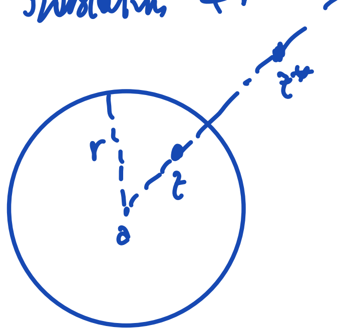
$$a^* = \frac{1}{a}$$



Using a rotation:



By translation & rotation



z^*, z on the same ray through the center

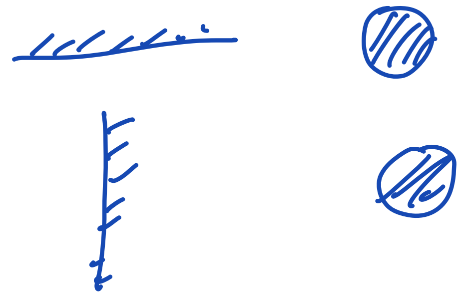
& $|z|\,|z^*| = r^2$

$$z^* = \frac{r^2}{\bar z - \bar a} + a.$$

Check Ahlfors!

$z \mapsto \dfrac{z-i}{z+i}$

$z \mapsto \dfrac{z-1}{z+1}$



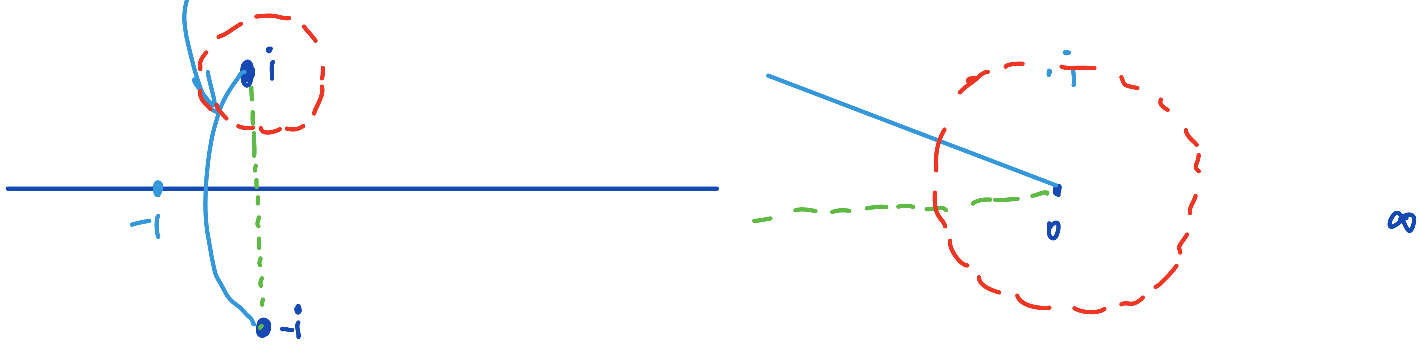
$\log z, \quad e^z, \quad z + \frac{1}{z}$

Ex

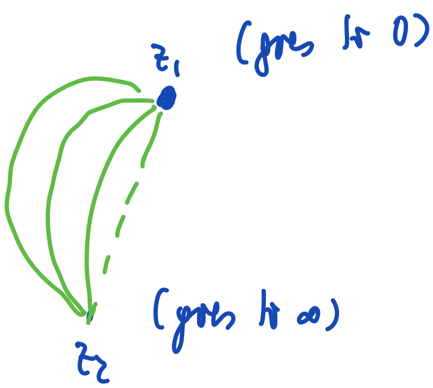
$z \longmapsto \dfrac{z-i}{z+i}$

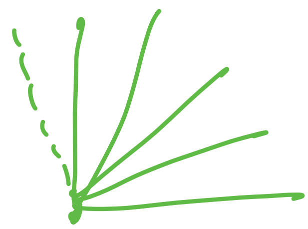
$$\frac{-1-i}{-1+i} = \frac{(-1-i)(-1-i)}{2} = i$$

they go to straight line



$z \mapsto \dfrac{az+b}{cz+d}$





COMPLEX INTEGRATION

Assume that $\gamma : [a,b] \to \Omega^{\text{open}} \subseteq \mathbb{C}$ is piecewise smooth.

Then for $f : \Omega \to \mathbb{C}$, we define

$$\int_\gamma f(z)\, dz = \underbrace{\int_a^b f(\gamma(t))\, \gamma'(t)\, dt}_{\text{usual 1D integral}}.$$

The right side should be divided into parts where γ is smooth.

We have invariance under a change of parameter on the smooth parts of γ.

Consider

$$t = t(\tau)$$

where

$$\tau : [\alpha, b] \to [a, b]$$

is piecewise smooth and γ is smooth.

Then we have

$$t \longmapsto \gamma(t)$$

$$t \mapsto \gamma(\tau(t))$$

Then

$$I = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt$$

Use the substitution $t \longmapsto \gamma(t)$. Let $t = \tau(s)$, $dt = \tau'(s)\, ds$

Then

$$I = \int_a^b f(\gamma(\tau(s)))\, \gamma'(\tau(s))\, \tau'(s)\, ds$$

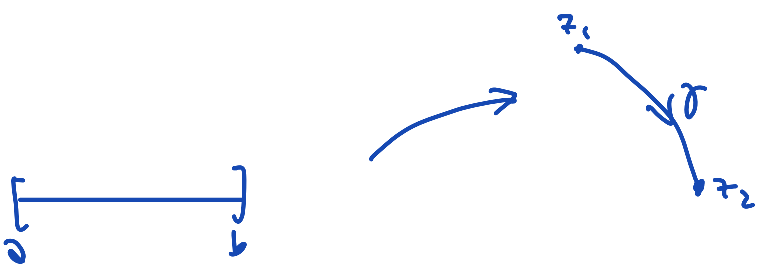
$$= \int_a^b f(\gamma(\tau(s)))\, (\gamma(\tau(s)))'\, ds$$

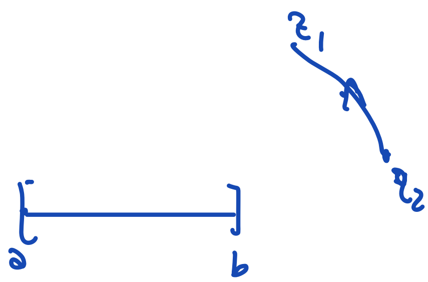
$$= \int_{\gamma \circ \tau} f(z)\,dz$$

Using reparametrization, we can prove that

$$\int_{-\gamma} f(z)\,dz = - \int_{\gamma} f(z)\,dz$$

where $-\gamma$ is $\gamma : [a,b] \to \mathbb{C}$ is defined as





We should consider all paths to be $\gamma : [0,1] \to \mathbb{C}$.
The paths defined for different a, b but having the path are considered the same.

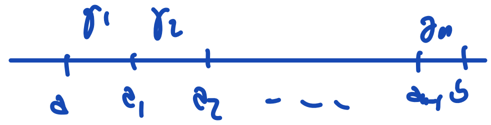
Assigned HW#2

Lecture 9
Mon, Feb 1

Also, if

$$\Gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$$

means:



assuming γ_{n-1} ends where γ_n starts, then

$$\int_{\gamma_1 + \dots + \gamma_n} f\,dz = \int_{\gamma_1} f\,dz + \dots + \int_{\gamma_n} f\,dz$$

Define integrals w.r.t. $\bar{z}$:

$$\int_\gamma f(z)\, d\bar{z} = \overline{\int_\gamma \overline{f(z)}\, dz}$$

Then, we can also define

$$\int_\gamma f\, dx = \frac{1}{2}\int_\gamma f\, dz + \frac{1}{2}\int_\gamma f\, d\bar{z}$$

$$\int_\gamma f\, dy = \frac{1}{2i}\int_\gamma f\, dz - \frac{1}{2i}\int_\gamma f\, d\bar{z}$$

Conway: defines $\int_\gamma f\, dz$ for curves of bdd variation ($\operatorname{Re}\gamma$, $\operatorname{Im}\gamma$ bdd variation). Such curves are called rectifiable.

Def $\gamma : [a,b] \to \Omega$ piecewise smooth. Then

$$\int_\gamma f\, |dz| = \int_\gamma f(z(t))\, |z'(t)|\, dt$$

is called the arc-length integral.

Closely related integrals

1) $\int_\gamma f \cdot d\vec{r}$ $\qquad d\vec{r} = (dx, dy)$

2) $\int_\gamma f\, ds$ $\qquad$ arc-length

can be complex valued

Thm Let p, q be cont. in $\Omega^{\text{open, conn}} \subseteq \mathbb{C}$.

Then

$$\int_\gamma (p\, dx + q\, dy)$$

depends only on the endpoints of γ iff

$\exists\, U \in C^1(\Omega)$ $\quad (U : \Omega \to \mathbb{C})$ s.t.

$$\frac{\partial U}{\partial x} = p \qquad\qquad (*)$$
$$\frac{\partial U}{\partial y} = q$$

Ω may be disconnected for (then the thm on each conn. comp.)

Proof ($\Leftarrow$) Assume $(*)$ holds for some U as in the statement, and let $(x(t), y(t))$ be a parametrization of γ on $[a,b]$. Then

$$\int_\gamma (p\,dx + q\,dy) \overset{(*)}{=} \int_\gamma \left(\frac{\partial U}{\partial x}(x,y)\,dx + \frac{\partial U}{\partial y}(x,y)\,dy\right)$$

$$\overset{\text{def'n of the line int.}}{=} \int_a^b \frac{\partial U}{\partial x} x'(t)\,dt + \int_a^b \frac{\partial U}{\partial y} y'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left(U(x(t), y(t)\right)\,dt$$

$$= U(x(b), y(b)) - U(x(a), y(a)).$$

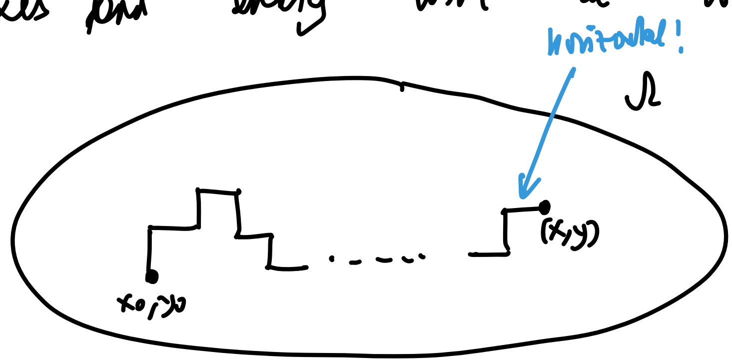
($\Rightarrow$) Assume $\int_\gamma (p\,dx + q\,dy)$ depends only on the endpoints.

Fix $(x_0, y_0) \in \Omega$, and define

$$U(x,y) = \int_\gamma (p\,dx + q\,dy)$$

where γ starts at (x_0, y_0) and ends at (x,y).

Consider the polygonal curve with segments parallel to the axes and ending with a horizontal segment:



Need: Ω open, conn.

Denote the last segment by $\overline{(x_1, y)\ (x, y)}$

So

$$U(x, y) = U(x_1, y) + \int_{x_1}^{x} p(t, y)\, dt$$

$$\Rightarrow \quad \frac{\partial U}{\partial x} = p(x, y)$$

Analogously (use a polygonal curve which ends vertically)

$$\frac{\partial U}{\partial y} = q(x, y).$$

Def Let $\Omega^{\text{open}} \subseteq \mathbb{C}$. We call $p\,dx + q\,dy$ (p, q $\mathbb{C}$-valued) is an exact differential if $\exists\, U \in C^1(\Omega)$ s.t.

$$\frac{\partial U}{\partial x} = p \quad \& \quad \frac{\partial U}{\partial y} = q.$$

Now let $f(z)$ be cont. and complex valued, and assume that $f(z)\, dz$ is an exact differential.

Recall:

$$f(z)\, dz = f(z)\, dx + i f(z)\, dy.$$

Then $\exists\, F \in C^1(\Omega, \mathbb{C})$ s.t.

$$\frac{\partial F}{\partial x} = f(z)$$

$$\frac{\partial F}{\partial y} = i f(z)$$

Observe that $\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y}$ ⟵ $\frac{\partial F}{\partial \bar z} = 0$ i.e. $\bar\partial F = 0$

the complex form of the C-R equations

Check (again): Write $F = U + iV$ (where U, V $\mathbb{R}$-valued)

$$F_x = U_x + i V_x$$
$$F_y = U_y + i V_y \quad \rightarrow \quad -i F_y = V_y - i U_y$$

Therefore,

$$\boxed{\begin{aligned} F_x = -iF_y \quad &\Leftrightarrow \quad U_x = V_y, \quad U_y = -V_x \\ &\Leftrightarrow \quad F \text{ analytic} \end{aligned}}$$

Recall: $F' = F_x$ $\qquad f' = u_x + iv_x = f_x$

Note: f needs to be analytic.

We have thus proven:

Thm Let $f \in C(\Omega)$, where $\Omega^{\text{open}} \subset \mathbb{C}$.
Then $\int_\gamma f\, dz$ depends only on endpoints iff
$\exists F$ analytic on Ω s.t.
$$F'(z) = f(z).$$

1. If f has a analytic primitive, we get $\int_{\gamma_1} f = \int_{\gamma_2} f$ independent of the integrity of f
2. If f is analytic then $\int_{\gamma_1} f \overset{?}{=} \int_{\gamma_2} f$ may fail

Lecture 10
Wed, Feb 3, 26

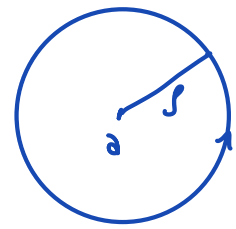
This says: $f(z)\,dz$ exact in $\Omega \Leftrightarrow f$ has an analytic primitive.

Corollary Let $n \in \mathbb{N}_0$. Then $\int_\gamma (z-a)^n\, dz = 0$ $\forall a \in \mathbb{C}$
and every closed curve γ in $\mathbb{C}$.

Proof: $(z-a)^n = \frac{d}{dz}\left(\frac{1}{n+1}(z-a)^{n+1}\right)$. ☒

Proposition Let C be a positively oriented circle around a w. radius ρ. Then

$$\int_C \frac{dz}{z-a} = 2\pi i \qquad (1)$$



and

$$\int_C \frac{dz}{(z-a)^m} = 0, \qquad m = 2, 3, \dots \qquad (2)$$

Proof (1): Use: $z = a + \rho e^{it}$ for $t \in [0, 2\pi]$.

Then

$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{\cancel{\rho} i e^{it}\, dt}{\cancel{\rho e^{it}}} = 2\pi i.$$

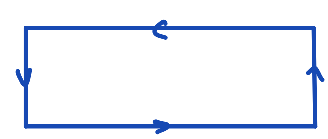
(2):

$$\frac{1}{(z-a)^m} = \frac{d}{dt}\left(\frac{1}{-m+1}(z-a)^{-m+1}\right)$$

($\leftarrow$ analytic in $D_\rho(a) \setminus \{a\}$)
$\{\gamma\}$ does not contain 0.
$\{\gamma\} \overset{def}{=} \{\gamma(t) : t \in [a,b]\}$. ∎

Theorem (Cauchy's Theorem for a rectangle)
Let $R = [a,b] \times [c,d]$, where $a < b$, $c < d$. If f is analytic on R, then

$$\int_{\partial R} f(z)\, dz = 0$$



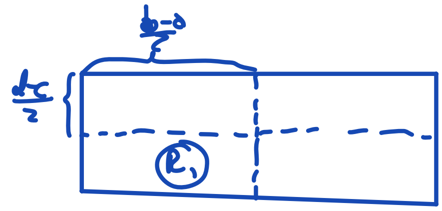
Recall: f analytic on R: $\exists\, \Omega^{open} \supseteq R$ and f is analytic in Ω.

We assume the positive orientation.

Proof For any rectangle $\tilde{R}$, denote

$$\eta(\tilde{R}) = \int_{\partial \tilde{R}} f(z)\, dz.$$

Divide R into four congruent rectangles



and select R_1 s.t.

$$|\eta(R_1)| \geq \frac{1}{4} |\eta(R)|$$

and continue by induction, obtaining

$$R \supseteq R_1 \supseteq R_2 \supseteq \dots$$

with

- R_{j+1} is one of the quarters of R_j
- $|\eta(R_{j+1})| \geq \frac{1}{4} |\eta(R_j)|$

The rectangles converge to a point z^*.
We use the differentiability of f at z^*.
Let $\varepsilon > 0$. Then $\exists\, \delta > 0$ s.t.

$$|z - z^*| \leq \delta \Rightarrow \left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| \leq \varepsilon$$

which is

$$|z - z^*| \leq \delta \Rightarrow |f(z) - f(z^*) - f'(z^*)(z - z^*)| \leq \varepsilon |z - z^*|.$$

Choose $n_0 \in \mathbb{N}$ s.t. $R_n \subseteq D_\delta(z^*)$ for $n \geq n_0$.
Then

DON'T: $\left|\int_\gamma f\right| \leq \int_\gamma |f| \neq 0$ $\leq \int_\gamma \max|f| = 0$

$$\eta(R_n) = \int_{\partial R_n} f(z)\, dz$$

$$= \int_{\partial R_n} \left(f(z) - f(z^*) - f'(z^*)(z - z^*) \right) dz$$

$$\left|\int_\gamma f\right| \le |\gamma| \sup_{\{\gamma\}} |f|$$

$\Rightarrow$
$$|\eta(R_n)| \le |\partial R_n| \sup_{\{\partial R_n\}} |f(z) - f(z^*) - f'(z^*)(z-z^*)|$$

$$\le \varepsilon\, |\partial R_n| \max_{\{\partial R_n\}} |z - z^*| \le \varepsilon\, |\partial R_n|^2$$

length of diag $\le$ perimeter

Therefore, for $n \ge n_0$

$$|\eta(R)| \le 4^n |\eta(R_n)| \le 4^n \varepsilon\, |\partial R_n|^2 = \cancel{4^n}\, \varepsilon\, |\partial R|^2 \frac{1}{\cancel{4^n}}$$

$\Rightarrow$
$$|\eta(R)| \le \varepsilon\, |\partial R|^2 .$$

Sending $\varepsilon \to 0$ gives $\eta(R) = 0$. ∎

We now prove a stronger thm.

Thm $R = [a,b] \times [c,d]$ rectangle $z_1, z_2, \dots, z_n \in R$.

Assume that f is analytic on

$$R' = R \setminus \{z_1, \dots, z_n\}$$

and assume

$$\lim_{z \to z_j} (z - z_j) f(z) = 0 \qquad (*)$$

a "removable singularity" condition

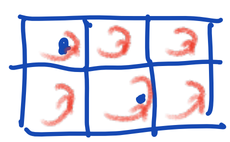
then
$$\int_{\partial R} f(z)\, dz = 0.$$

E.g.: f analytic in $R \setminus \{z_1, \dots, z_n\}$ and bdd in R (sufficient cond)

W/o $(*)$, the thm is not true: $\dfrac{1}{z - z_1}$.

idea:

Proof WLOG, $n = 1$ (subdivide R)

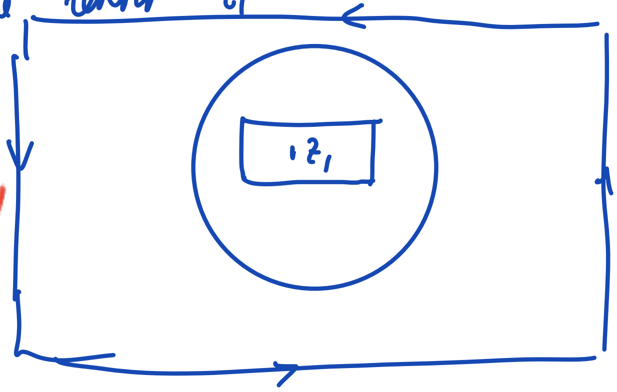


cancellation

Let $\varepsilon > 0$ be arbitrary. Then $\exists\, \delta > 0$ s.t.

$$|z - z_1| \le \delta \implies |f(z)| \le \frac{\varepsilon}{|z - z_1|}$$

Then find a square R_0 with the center z_1 contained in $D_\delta(z_1)$.



Then

$$\left| \int_{\partial R_0} f\,dz \right| \le \varepsilon \int_{\partial R_0} \frac{|dz|}{|z - z_1|}$$

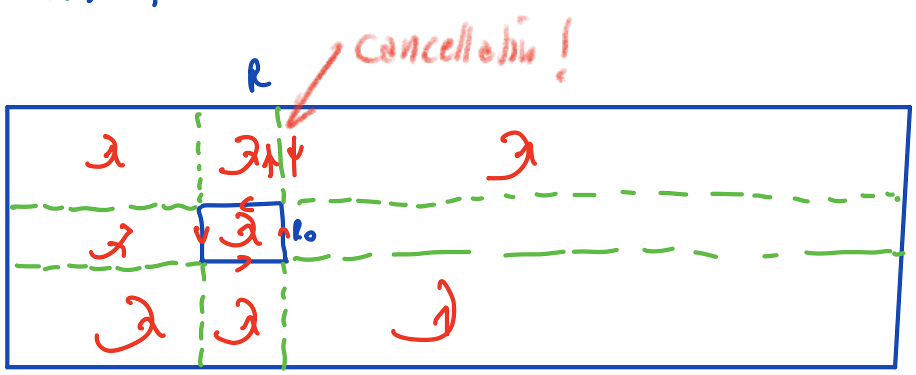
arc length

$$\le \frac{\varepsilon}{\min\limits_{z \in \partial R_0} |z - z_1|}\, |\partial R_0| \le \text{const } \varepsilon$$

Now use

$$\int_{\partial R_0} f\,dz = \int_{\partial R} f\,dz$$

To prove this, subdivide R as in the picture



$\therefore$

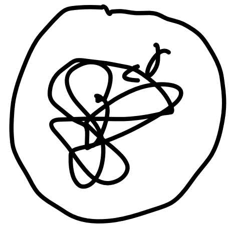
$$\left| \int_R f\,dz \right| = \left| \int_{R_0} f\,dz \right| \le \text{const } \varepsilon$$

Send $\varepsilon \to 0$.

Lecture 11
Fri, Feb 5, '26

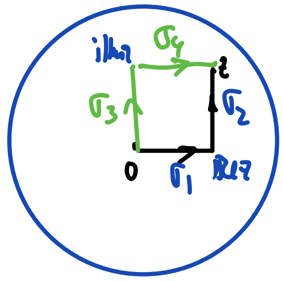
Theorem Assume that $f(z)$ analytic in $\mathbb{D}$, and γ is a closed piecewise smooth curve in $\mathbb{D}$. Then

$$\int_\gamma f(z)\,dz = 0. \qquad (*)$$



Later: will have this for simply-connected domains.

Proof Will use the theorem: $(*) \Leftrightarrow f$ has a primitive



Consider

$$F(z) = \int_{\sigma_1+\sigma_2} f\,dz$$

where σ_1 is the line segment from 0 to $\mathrm{Re}\,z$,
and σ_2 — " — " — $\mathrm{Re}\,z$ to z

Then

$$F(z) = \int_{\sigma_1} f(z)\,dx + i\int_{\sigma_2} f(z)\,dy$$

Then

$$\frac{\partial F}{\partial y} = if$$

we use: $\displaystyle\int_{\sigma_1+\sigma_2-\sigma_4-\sigma_3} f\,dz = 0 \Rightarrow \int_{\sigma_1+\sigma_2} f = \int_{\sigma_3+\sigma_4} f$

Note that

$$F(z) = \int_{\sigma_3+\sigma_4} f(z)\,dz \Leftarrow \text{by the Cauchy thm for rectangles}$$

where σ_3, σ_4 are as in the picture. Then

$$\frac{\partial F}{\partial x} = f$$

$$\therefore \quad \frac{\partial F}{\partial x} = -i\,\frac{\partial F}{\partial y}$$

$\Rightarrow F$ is analytic $\Rightarrow F'(z) = \dfrac{\partial F}{\partial x} = f.$

A generalization

<u>Thm</u> Let $z_1, z_2, \dots, z_n \in \mathbb{D}$ be distinct, and assume that f is analytic in $\mathbb{D} \setminus \{z_1, z_2, \dots, z_n\}$ with

$$\lim_{z \to z_j} (z - z_j) f(z) = 0 \quad , \quad j = 1, \dots, n$$

Then

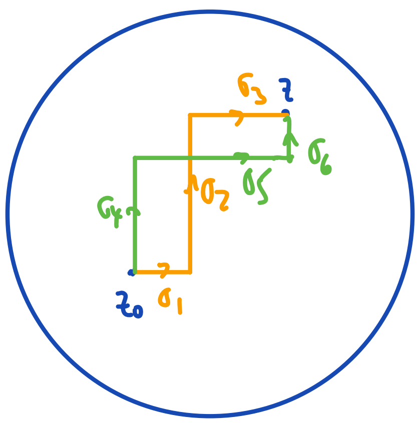
$$\int_\gamma f(z)\, dz = 0$$

for any closed piecewise smooth curve γ in Ω.

trick: change two segments to three segments.

FIXED

<u>Proof</u> Fix $z_0 \in \mathbb{D}$. For $z \in \mathbb{D} \setminus \{z_1, \dots, z_n\}$, choose a path $\sigma_1 + \sigma_2 + \sigma_3$ as in the picture - we require $z_1, \dots, z_n \notin \{\sigma_1\} \cup \{\sigma_2\} \cup \{\sigma_3\}$



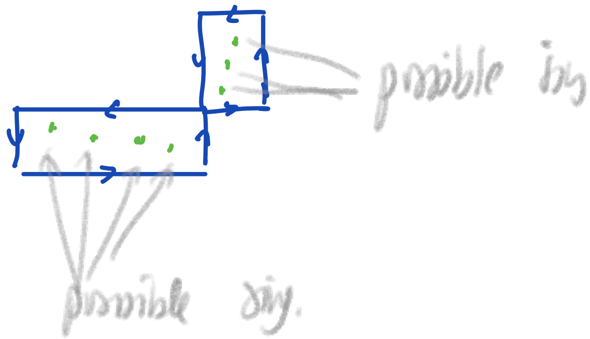
Define

$$F(z) = \int_{\sigma_1 + \sigma_2 + \sigma_3} f(z)\, dz$$

Note that the definition does not depend on the choice of $\sigma_1, \sigma_2, \sigma_3$ (the Thm on rectangles).

Then
$$\frac{\partial F}{\partial x} = f$$

Choosing $\sigma_4, \sigma_5, \sigma_6$ as in the picture, making sure that $z_1, \dots, z_n \notin \{\sigma_4\} \cup \{\sigma_5\} \cup \{\sigma_6\}$, we have
$$F(z) = \int_{\sigma_4+\sigma_5+\sigma_6} f \, dz$$
The difference is:



Then
$$\frac{\partial F}{\partial y} = i f$$
Therefore, f has a primitive in $D \setminus \{z_1, \dots, z_n\}$. $\blacksquare$

INDEX (WINDING NUMBER)

To define the index of a curve around a point, we need:

Lemma Let $a \in \mathbb{C}$, and let γ be a piecewise smooth closed path not passing through a $(a \notin \{\gamma\})$. Then
$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}.$$

Recall: γ is oriented

Def For $a \notin \{\gamma\}$ ($\{\gamma\}$ ... image of $\gamma : [a,b] \to \mathbb{C}$)

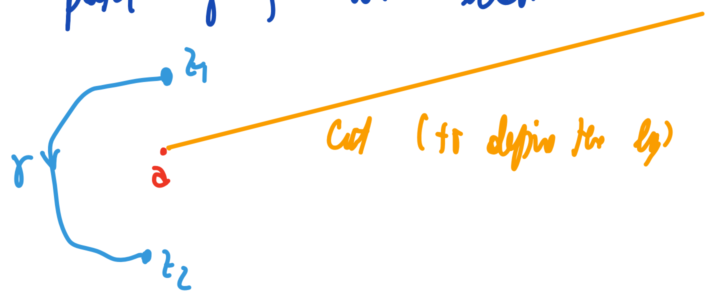
define

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

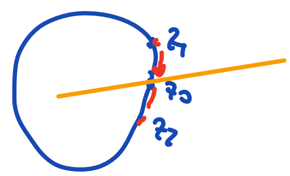
and call it index (or a winding number)

Motivation:

If log exists on some path γ, we have



$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\log(z-a)\Big|_{z_1}^{z_2}$$



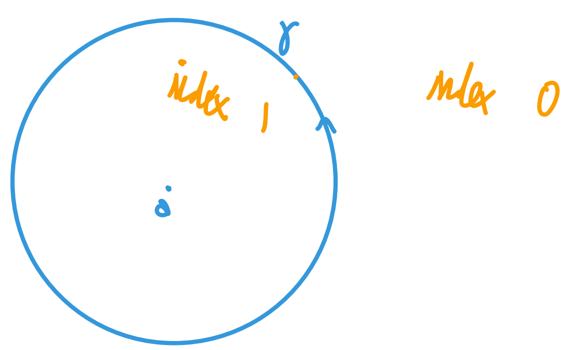
$$= \frac{1}{2\pi i}\log\frac{|z_2-a|}{|z_1-a|} + \frac{1}{2\pi}(\arg z_2 - \arg z_1)$$

where $z_1 = \gamma(a)$, $z_2 = \gamma(b)$.

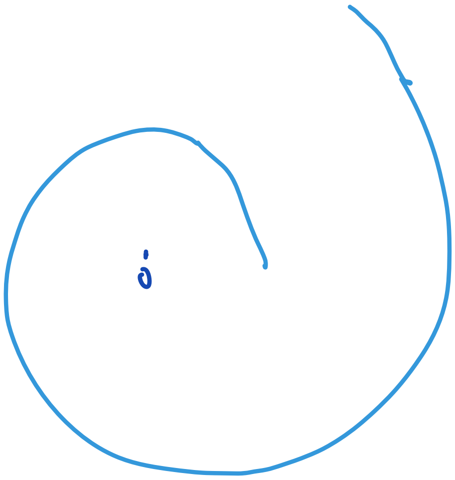
As $z_2 \to z_1$ (the curve is closing)

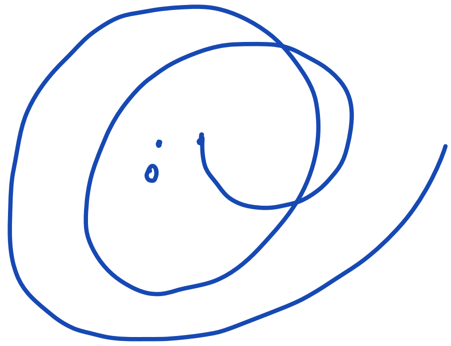
the first part approaches 0 and second approaches the index.

Exercise Compute the index of a circle around points inside and outside of the circle



Lecture 12
Mon, Feb 8





$0=0$ for simplicity

Consider a curve $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ and a fn

$$\tilde{\theta} : [0,1] \to \mathbb{R} / 2\pi\mathbb{Z}$$

s.t. $\tilde{\theta}(t)$ is the argument of $\gamma(t)$.

<u>Prop</u> There exists a unique, up to a constant in $2\pi\mathbb{Z}$, continuous fn

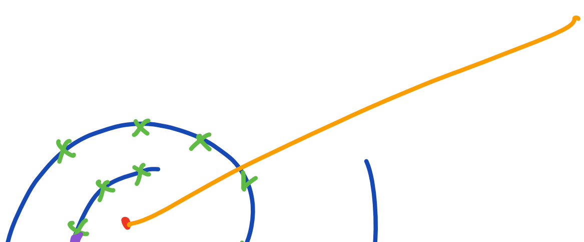
$$\theta : [0,1] \to \mathbb{R}$$

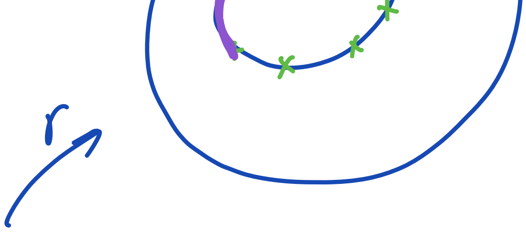
s.t.

$$\theta(t) \in \tilde{\theta}(t) , \qquad \forall t \in [0,1]$$

Says: We can assign a continuous argument on the curve γ. It is unique up to a multiple of 2π

<u>Proof</u> exercise: Idea: We define θ by <u>intervals</u> where we can define the argument uniquely





Note: We cannot define a cont. argument on $\mathbb{C} \setminus \{0\}$
but we can define it on a curve.

$$\mathrm{ind}_0 \gamma = \theta(1) - \theta(0)$$

Second definition: Based on

classical log

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i} \log \frac{\overset{\text{zero}}{|z_2 - a|}}{\underset{\text{zero}}{|z_1 - a|}} + \frac{1}{2\pi}\left(\arg z_2 - \arg z_1\right)$$

for $z_1 = \gamma(0)$, $z_2 = \gamma(1)$.

So

$$\mathrm{ind}_0 \gamma = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} - \frac{1}{2\pi i} \log \frac{|\gamma(1) - 0|}{|\gamma(0) - 0|}$$

Proof Let $z(t)$, for $\alpha \le t \le \beta$, be a parametrization of γ. Let

$$h(t) = \int_\alpha^t \frac{z'(s)}{z(s) - a}\, ds$$

Difficulty: log is not uniquely defined.
Trick: Consider $e^{h(t)}$. We claim

$$\left(e^{-h(t)}\left(z(t) - a\right)\right)' = 0$$

(Idea: Since we "expect" $h(t) = \log (z(t)-a) \Big|_{\alpha}^{t}$

$= \log \frac{z(t)-a}{z(\alpha)-a}$

(not displayed); but $e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}$ $\therefore$ $e^{-h(t)} = \frac{z(\alpha)-a}{z(t)-a}$

$\therefore$ $(z(t)-a)\, e^{-h(t)} = z(\alpha)-a$ ... constant!) $\leftarrow$ derivative in $t \in \mathbb{R}$

Claim: $(z(t)-a)\, e^{-h(t)}$ is constant

Proof. $\left(e^{-h}(z(t)-a)\right)' = -h'(t)\, e^{-h(t)} (z(t)-a) + e^{-h(t)} z'(t)$

$= -\frac{z'(t)}{\cancel{z(t)-a}}\, e^{-h(t)}\, \cancel{(z(t)-a)} + e^{-h(t)} z'(t)$

$= 0$ ※

(Note that $e^{-h(t)}(z(t)-a)$ is piecewise C' with the derivative zero — so it is constant.)

By the claim:

$$(z(t)-a)\, e^{-h(t)} = (z(\alpha)-a)\underbrace{e^{-h(\alpha)}}_{=1} = z(\alpha)-a$$

$$\Rightarrow \quad e^{-h(t)} = \frac{z(\alpha)-a}{z(t)-a}$$

Substitute $t=\beta$:

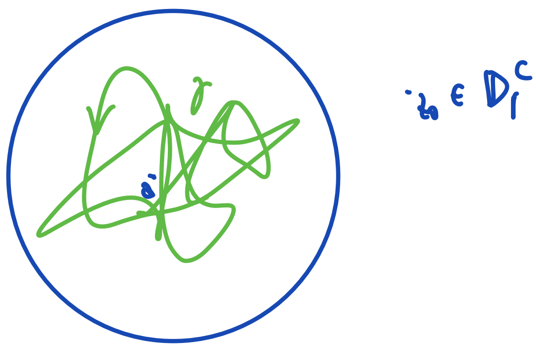
$$e^{-h(\beta)} = \frac{z(\alpha)-a}{z(\beta)-a} = 1$$

$\therefore$ $h(\beta) \in 2\pi i \mathbb{Z}$.

$\Rightarrow$ $\frac{1}{2\pi i} h(\beta) \in \mathbb{Z}$ as claimed. $\square$

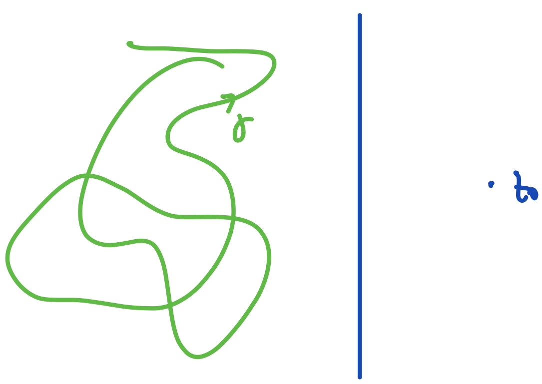
Proposition Assume that a closed path γ is such that $\{\gamma\} \subseteq \mathbb{D}_r(a)$, for some $a \in \mathbb{C}$ and $r > 0$. Then

$$n(\gamma, z_0) = 0 \qquad \forall z_0 \in \mathbb{D}_r(a)^c.$$

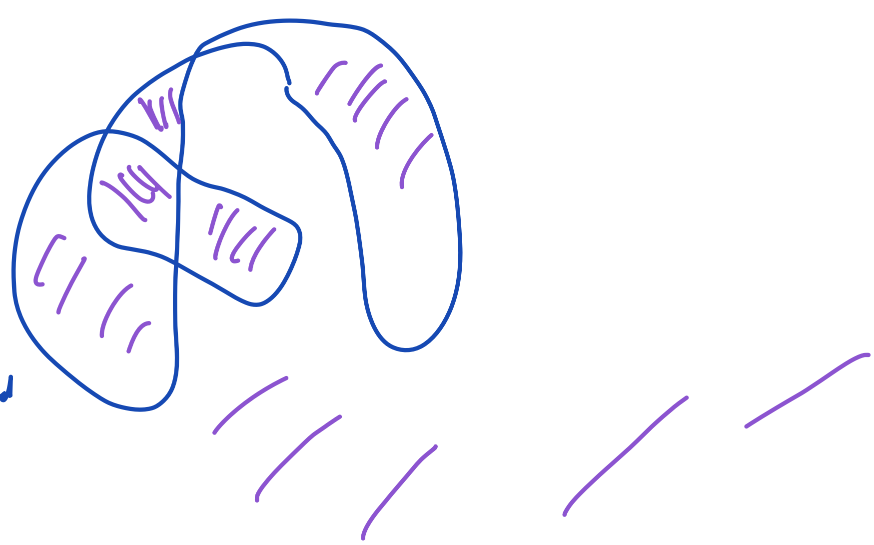


Proof $\frac{1}{z-z_0}$ is analytic in D_r. So $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} = 0$. $\blacksquare$

The proof also works for half space



Let γ be a closed path. Then $\mathbb{C} \setminus \{\gamma\}$ is open so it is a union of open connected sets.



These components are called regions determined by γ.

Ex There could be infinitely many regions
- exercise. Use the function

$$\gamma(t) = (t-\tfrac{1}{2})^2 \sin\left(\frac{1}{t-1/2}\right) \quad , \quad t \in [0, \tfrac{1}{2})$$

and similarly for $t \in [\tfrac{1}{2}, 1]$.

Ex Is there an elementary proof of the Jordan curve theorem!

Proposition The index $n(\gamma, a)$ is constant in each region determined by γ and it is zero in the unbdd region.

Proof The index is continuous in $\mathbb{C} \setminus \{\gamma\}$ and integer valued.
For the unbdd region apply the prev. prop.

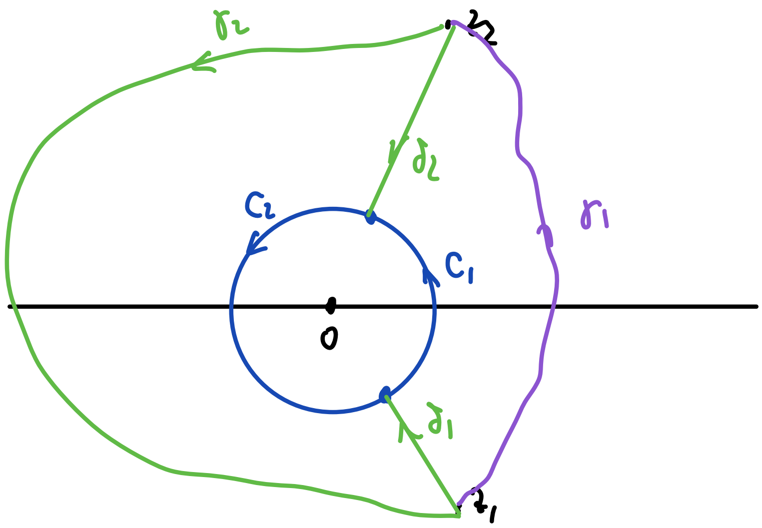
Lemma Let γ be a closed path s.t. $0 \notin \{\gamma\}$, and let $z_1, z_2 \in \{\gamma\}$. Let γ_1 be the part of γ going from z_1 to z_2, and γ_2 be the part of γ going from z_2 to z_1.
Suppose that $\operatorname{Im} z_2 > 0 > \operatorname{Im} z_1$. Assume

$$\gamma_1 \cap (-\infty, 0) = \emptyset$$

$$\gamma_2 \cap (0, \infty) = \emptyset$$

Then

$$n(\gamma, 0) = 1.$$



Lecture 13
Wed, Feb 10

Proof Let δ_1, δ_2 be paths from z_1, z_2 to C resp, where C is a circle around the origin,

$$C = C_1 + C_2$$

where C_1, C_2 are as in the picture

Consider

$$\sigma_1 = \gamma_1 + \delta_2 - C_1 - \delta_1 \qquad \text{(closed path)}$$
$$\sigma_2 = \gamma_2 + \delta_1 - C_2 - \delta_2 \qquad 〃$$

Then

$$\gamma = \gamma_1 + \gamma_2$$
$$= (\sigma_1 \cancel{- \delta_2} + C_1 \cancel{+ \delta_1}) + (\sigma_2 \cancel{- \delta_1} + C_2 \cancel{+ \delta_2})$$

$\therefore$

$$\gamma = \sigma_1 + \sigma_2 + C$$

$\Rightarrow$

$$n(\gamma, 0) = \underbrace{n(\sigma_1, 0)}_{\because\ 0 \text{ is in the unbdd region of } \sigma_1} + \underbrace{n(\sigma_2, 0)}_{=0\ \because\ 0 \text{ is in the unbdd region of } \sigma_2} + n(C, 0)$$

$$= 0 + 0 + 1$$

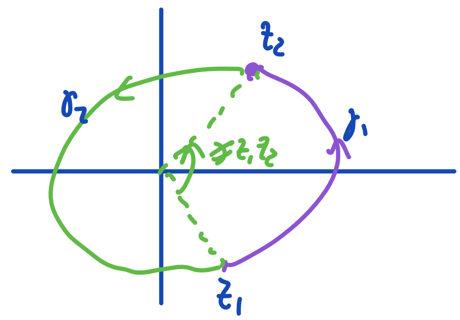
$\Rightarrow$

$$n(\gamma, 0) = 1.$$

∎

Remark A "direct" proof: To define on the argument on γ_1, we use the cut $(-\infty, 0)$. The argument increases by

$$\measuredangle z_1 z_2$$

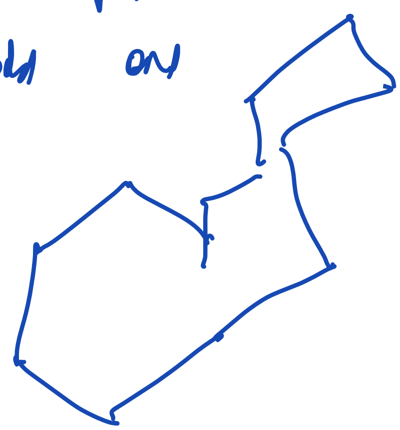


To define the argument on γ_2, we cut $(0, \infty)$, the argument increases by $2\pi - \measuredangle z_1 z_2$.
The total increase of $\gamma_1 + \gamma_2$ is

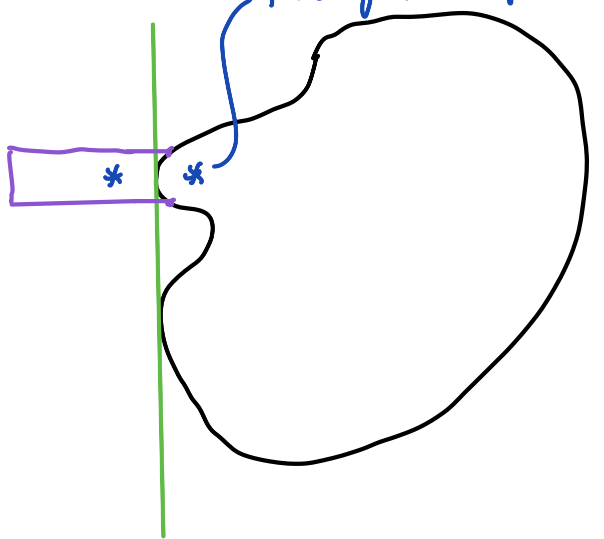
$$2\pi$$

So the index is 1. ✳

Ahlfors: A <u>piecewise smooth</u> Jordan curve (closed path without self-intersection) divide a plane into <u>at least</u> two components, one unbdd one at least one bdd with 1 or -1.



Ahlfors proof by picture
First rotate & translate, so we get:
index of this pt is 1



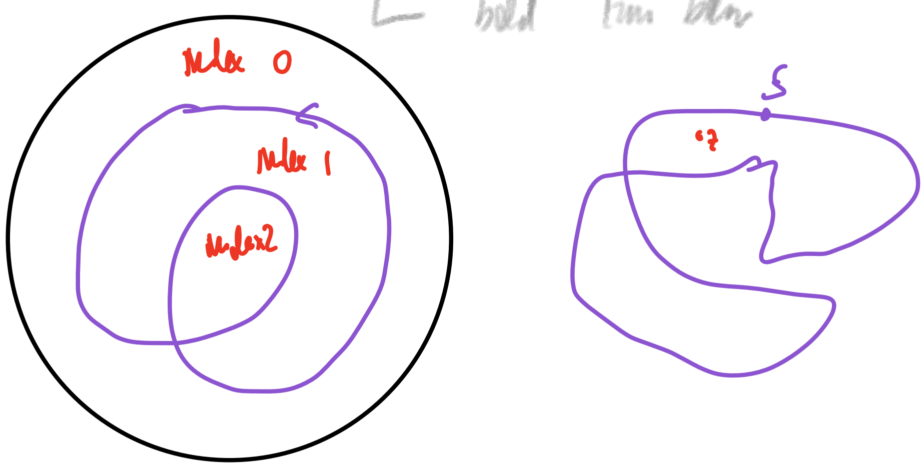
CAUCHY INTEGRAL FORMULA

A local Cauchy integral formula.

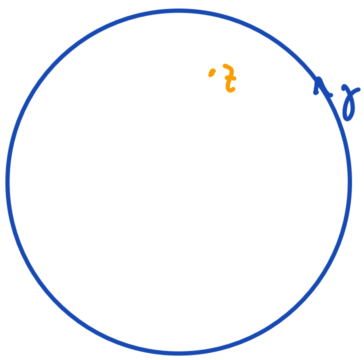
Theorem Assume that f is analytic in a disc $D = \mathbb{D}_r(a)$, where $r>0$, and let γ be a closed path in D.
Then

$$n(\gamma, z)\, f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)\, d\xi}{\xi - z}, \quad z \in D \setminus \{\gamma\}.$$

hold from below



Most of the time, we use the Cauchy integral formula with a circular path with z not necessarily in the center



Proof Consider

$$F(\xi) = \frac{f(\xi) - f(z)}{\xi - z}, \quad \xi \in D \setminus \{z\}$$

Then the fn. F is analytic (in ξ) in $D \setminus \{z\}$, and at z, we have

$$\lim_{\xi \to z} (\xi - z) F(\xi) = \lim_{\xi \to z} (f(\xi) - f(z)) = 0$$

Therefore, F satisfies the conditions of the generalized version of the Cauchy thm. Since $z \notin \{\gamma\}$, we have

$$\int_\gamma F(\xi)\, d\xi = 0$$

$\Rightarrow$ (using $\xi \neq z$ $\forall \xi$)

$$\int_\gamma \frac{f(\xi) - f(z)}{\xi - z}\, d\xi = 0$$

$$\Rightarrow \quad \int_\gamma \frac{f(z)}{\xi - z}\, d\xi = \int_\gamma \frac{f(\xi)}{\xi - z}$$

$$\Rightarrow \quad f(z) \underbrace{\int_\gamma \frac{d\xi}{\xi - z}}_{= n(\gamma, z)} = \int \frac{f(\xi)}{\xi - z}. \qquad \blacksquare$$

always

Theorem Assume that f is analytic in Ω open $(\neq \emptyset)$, let $\overline{D_r(a)} \subseteq \Omega$, where $a \in \mathbb{C}$, $r > 0$. Then f is infinitely (complex) differentiable in $D = D_r(a)$ and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi, \qquad z \in D.$$

(D is not the same as $\mathbb{D} = D_1(0)$)

$$f^{(n)}(z) \overset{\text{def}}{=} \frac{d^n f}{dz^n}(z)$$

Lecture 14
Wed, Feb 12