Math 520 Homework 11

Due: Friday, May 1, 2026, 7pm

The homework is optional—no credit.

The homework can be turned in as a pdf to the gradescope; code 5DN86V

1. [Ahlfors, p. 186, #1] Prove that the Laurent development is unique.

2. [Ahlfors, p. 186, #3] The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the Schwarzian derivative of f. If f has a multiple zero or pole, find the leading term in the Laurent development of $\{f, z\}$. (Answer: If $f(z) = a(z - z_0)^m + \cdots$, then $\{f, z\} = \frac{1}{2}(1 - m^2)(z - z_0)^{-2} + \cdots$.)

3. [Ahlfors, p. 193, #1] Show that

$$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

4. [Ahlfors, p. 193, #3] Prove that

$$\prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges absolutely and uniformly on every compact set.

5. [Conway, p. 127, #8] Is a non-constant meromorphic function on a region G an open mapping of G into $\mathbb{C}$? Is it an open mapping of G into $\mathbb{C}_\infty$?

6. [Conway, p. 127, #9] Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \operatorname{Re} z > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \to 1$?