Math 520 Homework 2

Due: Monday, February 9, 2026
The homework is optional—no credit.
The homework needs to be turned in as a pdf to the gradescope; code 5DN86V

Read [Alhfors, pp. 28–32] on polynomials and rational functions.

1. [Alhfors, p. 32 #2] If Q is a polynomial with distinct roots $\alpha_1, \ldots, \alpha_n$, and if P is a polynomial of degree $< n$, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)}.$$

2. [Alhfors, p. 32 #3] Use the formula in the preceding exercise to prove that there exists a unique polynomial of degree $< n$ with given values c_k at the points α_k (Lagrange's interpolation polynomial).

3. What is the general form of a rational function which maps $\mathbb{R}$ to $\mathbb{R}$? In particular, how are the zeros and poles related to each other? (Hint: If R is the rational function, consider the difference $R(z) - \overline{R(\overline{z})}$.)

4. [Alhfors, p. 33 # 4] What is the general form of a rational function which has absolute value 1 on the circle $|z| = 1$? In particular, how are the zeros and poles related to each other? (Hint: If R is the rational function, consider the product $R(z) \cdot \overline{R(1/\overline{z})}$.)

5. [Alhfors, p. 78 # 1] Prove that the reflection $z \mapsto \overline{z}$ is not a fractional linear transformation.

6. [Alhfors, p. 78 #3] Prove that the most general transformation which leaves the origin fixed and preserves all distances is either a rotation or a rotation followed by reflection in the real axis.

7. [Conway, p. 54 #1] Find the image of $\{z \in \mathbb{C} : \mathbb{R}\mathrm{e}\, z < 0, |\,\mathbb{I}\mathrm{m}\, z| < \pi\}$ under the exponential function.