Math 520 Homework 8

Due: Monday, April 6, 2026, 7pm
The homework is optional—no credit.
The homework can be turned in as a pdf to the gradescope; code 5DN86V

1. 1362 If $f(z)$ is analytic and $\mathbb{I}\mathrm{m}\, f(z) > 0$ for $\mathbb{I}\mathrm{m}\, z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$$

and

$$\frac{|f'(z)|}{\mathbb{I}\mathrm{m}\, f(z)} \leq \frac{1}{y}$$

($z = x + iy$).

2. 1365 Prove by use of Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a linear transformation.

3. [Conway, p. 132, #1] Suppose $|f(z)| \leq 1$ for $|z| < 1$ and f is non-constant analytic function. By considering the function $g\colon \mathbb{D} \to \mathbb{D}$ defined by

$$g(z) = \frac{f(z) - a}{1 - \overline{a} f(z)},$$

where $a = f(0)$, prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$

for $|z| < 1$.

4. [Conway, p. 132, #2] Does there exist an analytic function $f\colon \mathbb{D} \to \mathbb{D}$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$.

5. [Conway, p. 133, #5] Let f be analytic in $\mathbb{D}$, and suppose that $|f(z)| \leq M$ for all $z \in \mathbb{D}$.
(a) If $f(z_k) = 0$ for $1 \leq k \leq n$ show that

$$|f(z)| \leq M \prod_{k=1}^{n} \frac{|z - z_k|}{|1 - \overline{z}_k z|}$$

for $|z| < 1$.
(b) If $f(z_k) = 0$ for $1 \leq k \leq n$, each $z_k \neq 0$, and $f(0) = Me^{ia} z_1 z_2 \cdots z_n$, find a formula for f.

6. [Conway, p. 133, #6] Suppose f is analytic in some region containing $\overline{\mathbb{D}}$ and $|f(z)| = 1$ for $|z| = 1$. Find a formula for f. (Hint: First consider the case where f has no zeros in $\mathbb{D}$.)

7. [Conway, p. 133, #8] Is there an analytic function f on $\mathbb{D}$ such that $|f(z)| < 1$ for $|z| < 1$, $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$? If so, find such an f. Is it unique?