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TVERBERG'S PROOF OF THE JORDAN CLOSED CURVE THEOREM

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Abstract. A proof of the classical theorem on a simple closed curve (Jordan's theorem) is discussed; this proof is given by a Norwegian mathematician H. Tverberg and is little known to specialists. The proof has a metric nature and makes it possible to obtain an important metric refinement of Jordan's theorem, which is interesting on its own.

§1. Introduction

The following fundamental topological fact is known as the *Jordan closed curve theorem* or as *Jordan's theorem* [1].

Jordan's Theorem. *Let* $\mathbb{T} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ *denote the unit circle in* $\mathbb{R}^2$*, and let* $\gamma\colon \mathbb{T} \to \mathbb{R}^2$ *be a continuous injective map, i.e.,* $\Gamma = \gamma(\mathbb{T})$ *is a simple closed curve on the plane. Then the set* $\mathbb{R}^2 \setminus \Gamma$ *has exactly two connected components* (*disjoint domains*).

This theorem plays a special role also in analysis. For many years, when lecturing general and special courses in complex analysis at Lomonosov Moscow State University's Department of mathematics and mechanics, and in Bauman Moscow State Technical University's Department of fundamental sciences, the authors of the present note have been searching for an adequately simple and complete proof of Jordan's theorem. Such a proof was found in the paper [2] of the well-known Norwegian mathematician H. Tverberg. After substantial revision and adding details, it was included in our special courses for several times. We think that it is also accessible to high school students who are familiar with a basic calculus course (notions of limit and continuity, the Bolzano–Weierstrass Theorem, the Nested Intervals Theorem). Note that this proof allows us to obtain an important *metric* refinement of Jordan's theorem, which is interesting on its own (see Corollary 1).

§2. Introductory remarks and auxiliary lemmas

Here we collect several elementary facts from analysis that will be used in what follows. First, note that in the above notation, the map γ is uniformly continuous on $\mathbb{T}$, and the inverse map $\gamma^{-1}\colon \Gamma \to \mathbb{T}$ is also continuous. Next, if A and B are nonempty disjoint compact sets in $\mathbb{R}^2$, then

$$\mathrm{d}(A, B) := \inf\{|a - b| : a \in A,\ b \in B\}$$

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is greater than zero. The proof of these claims is based on the Bolzano–Weierstrass theorem, which states that each bounded sequence of real numbers has a convergent subsequence.

Recall also that a *domain* in $\mathbb{R}^2$ is any nonempty open set such that every two points in it can be connected by a polygonal line lying inside that domain.

In what follows we assume that the unit circle $\mathbb{T}$ is oriented counter-clockwise in a way consistent with the natural parametrization $t(\theta) = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$.

The proposed proof of Jordan's theorem is based on a special approximation of the curve Γ by simple closed polygonal lines, and then passing to the limit. This natural approach is well known, so that Lemmas 1 and 2 below are not new. On the other hand, Lemma 3 and Lemma 4 are new, and are interesting on their own. They are aimed at obtaining a certain *metric* description of the simple closed polygonal lines mentioned above, which helps to evaluate the limit. The main problem is to avoid the situation that can happen for a (not simple) closed curve, which looks like the symbol ∞ and can be expressed (for a certain parametrization) as the limit of simple closed polygonal lines.

Definition 1. A simple closed curve $\Sigma = \sigma(\mathbb{T})$, where $\sigma\colon \mathbb{T} \to \Sigma$ is a homeomorphism, is called a *simple close polygon* if there exists a partition $\Theta = \{\theta_0, \theta_1, \dots, \theta_N\}$ of the interval $[0, 2\pi]$ (i.e., $0 = \theta_0 < \theta_1 < \dots < \theta_N = 2\pi$) such that

$$\sigma\big((\cos\theta, \sin\theta)\big) = (a_n\theta + b_n, c_n\theta + d_n)$$

on $[\theta_{n-1}, \theta_n]$ for $n = 1, \dots, N$, where a_n, b_n, c_n, and d_n are real constants.

Remark. In accordance with the choice of a partition Θ, we can define *vertices* and *edges* of a polygon Σ in a natural way. Note that adjacent edges of a polygon Σ can lie on the same line.

A pair (σ, Θ) is called a *realization* of a polygon Σ.

Lemma 1. *Jordan's theorem is true for any simple closed polygon.*

Proof. Let Σ be a simple closed polygon, and let $v_n = \sigma((\cos\theta_n, \sin\theta_n))$ and $\Sigma_n = [v_{n-1}, v_n]$ for $n = 1, \dots, N$ be its vertices and edges. Assume also that $\Sigma_{N+1} = \Sigma_1$ and $v_0 = v_N$.

First, we prove that $\mathbb{R}^2 \setminus \Sigma$ has *at most two* connected components. We treat the case where $N \geq 4$. For every $n = 1, \dots, N$, consider the set $U_n = \{z \in \mathbb{R}^2 \mid \mathrm{d}(z, \Sigma_n) < \delta\}$, where

$$\delta = \frac{1}{2}\min\{\mathrm{d}(\Sigma_j, \Sigma_k)\}$$

and the minimum is taken over all *nonadjacent* edges of Σ. If we denote $\Sigma_0 = \Sigma_N$, then, clearly,

$$U_n \cap \Sigma \subset \Sigma_{n-1} \cup \Sigma_n \cup \Sigma_{n+1},$$

and $U_n \setminus \Sigma$ consists of two components U_n' and U_n'', where we may assume for convenience that $U_n' \cap U_{n+1}' \neq \varnothing$ and $U_n'' \cap U_{n+1}'' \neq \varnothing$ for $n = 1, \dots, N-1$. The sets $U' := \bigcup_{n=1}^N U_n'$ and $U'' := \bigcup_{n=1}^N U_n''$ are domains, and every point $z \in \mathbb{R}^2 \setminus \Sigma$ can be connected by a line segment to either U' or U'' outside of Σ.

Now we prove that $\mathbb{R}^2 \setminus \Sigma$ has *at least two* connected components. We choose a coordinate system in $\mathbb{R}^2$ in such a way that all the vertices $v_n = (x_n, y_n)$ of the polygon Σ have distinct x-coordinates x_n.

For any $z \notin \Sigma$, we put $\eta(z) = 1$ if the ray L_z with initial point z that goes upright intersects Σ an *odd* number of times. For an even number of intersections of L_z and Σ, we define $\eta(z) = 0$. Note that if L_z contains (exactly one) vertex, say, v_k, of the polygon Σ, and if the edges Σ_{k-1} and Σ_k (which intersect at the point v_k) lie on the same side

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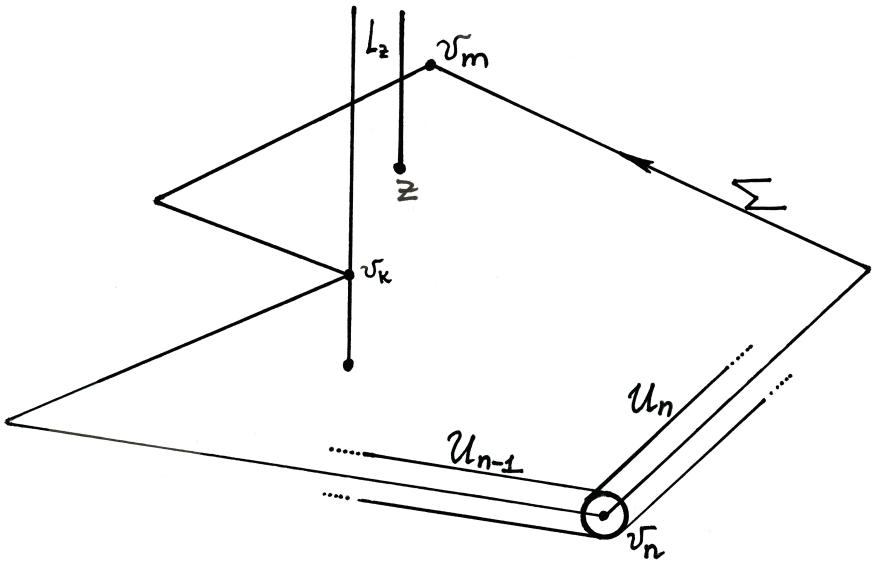

FIGURE 1. To the proof of Lemma 1.

of L_z, then we postulate that in a neighborhood of the point z the ray L_z has two (or zero) intersections with Σ (see Figure 1). It is easy to prove that the function $\eta(z)$ is continuous at every point outside of Σ, and, since it is integer-valued, it is a locally constant function of z outside of Σ. It follows that $\eta(z)$ is constant on every connected component of the set $\mathbb{R}^2 \setminus \Sigma$.

Suppose that the set $\mathbb{R}^2 \setminus \Sigma$ has only one (unbounded) connected component; it is obvious that then $\eta(z)$ is identically zero. Suppose now that $v_m = (x_m, y_m)$ is a vertex of Σ such that $y_m = \max\{y_n : n = 1, \dots, N\}$. Then, clearly close to the point v_m there exists a point z such that $\eta(z) = 1$. □

Lemma 2. *Any simple closed path γ (recall that $\gamma\colon \mathbb{T} \to \Gamma$ is a homeomorphism) can be uniformly on $\mathbb{T}$ approximated with an arbitrary accuracy by a simple closed path σ that determines a simple closed polygon Σ as in Definition 1. Moreover, Σ is "inscribed" into Γ in the sense that all its vertices lie on Γ.*

Proof. Fixing an arbitrary $\varepsilon > 0$, we choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the following conditions are satisfied for every $t \in \mathbb{T}$, $t' \in \mathbb{T}$:

(1) if $|t - t'| \le \varepsilon_1$, then $|\gamma(t) - \gamma(t')| < \frac{\varepsilon}{2}$;
(2) if $|\gamma(t) - \gamma(t')| \le \varepsilon_2$, then $|t - t'| < \min\left(\varepsilon_1, \sqrt{3}\right)$.

Now we put $\delta = \min\{\varepsilon/2, \varepsilon_2\}$ and consider a standard lattice of (closed) squares of diameter δ:

$$Q_{jk} = \left\{(x, y) : \left|x - j\frac{\delta}{\sqrt{2}}\right| \le \frac{\delta}{2\sqrt{2}}, \ \left|y - k\frac{\delta}{\sqrt{2}}\right| \le \frac{\delta}{2\sqrt{2}}\right\}, \quad j, k, \in \mathbb{Z}.$$

Let $\{Q_s\}_{s=1}^{S}$ be the set of squares that intersect Γ at more than one point. It is easily seen that $2 \le S < +\infty$. Since $\delta \le \varepsilon_2$, the set $\gamma^{-1}(Q_1)$ has diameter less than $\sqrt{3}$, i.e., $\gamma^{-1}(Q_1)$ is contained in a (uniquely determined) minimal closed arc $T_1 \subset \mathbb{T}$ of length less than $2\pi/3$. Let $[\tau_1, \tau_1'] \subset \mathbb{R}$ be an interval such that $\theta \mapsto (\cos\theta, \sin\theta)$ is a homeomorphism between $[\tau_1, \tau_1']$ and T_1. Consider a new path γ_1 on $\mathbb{T}$ that coincides with γ on $\mathbb{T}\setminus T_1$, while

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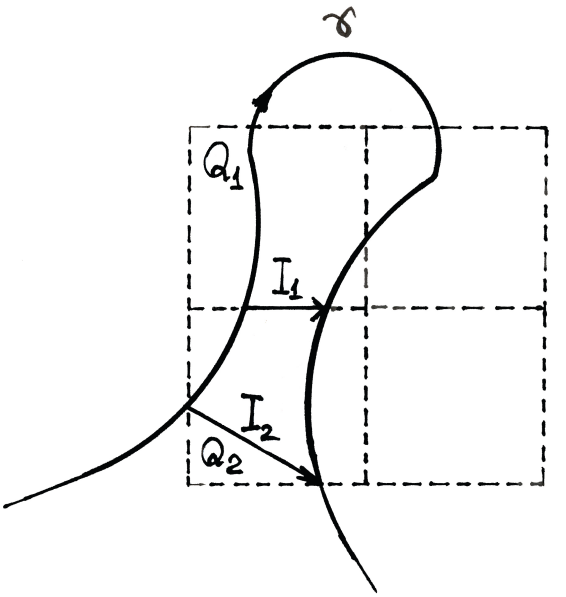

FIGURE 2. To the proof of Lemma 2: Q_1 — case (i), Q_2 — case (ii).

for $t = (\cos\theta, \sin\theta) \in T_1$ (when $\theta \in [\tau_1, \tau_1']$) we put $\gamma_1(t) = (a_1\theta + b_1, c_1\theta + d_1)$, where the constants a_1, b_1, c_1, d_1 are chosen in such a way that γ_1 is continuous on $\mathbb{T}$. Therefore, $\Gamma_1 = \gamma_1(\mathbb{T})$ has a common line segment $[\gamma(t_1), \gamma(t_1')]$ with Q_1, where $t_1 = (\cos\tau_1, \sin\tau_1)$ and $t_1' = (\cos\tau_1', \sin\tau_1')$ are the beginning and the end of the arc T_1, respectively. It is clear that $\gamma_1(t_1) = \gamma(t_1)$ and $\gamma_1(t_1') = \gamma(t_1')$.

Two cases are possible.

In the first case (i) the interval $I_1 = [\gamma_1(t_1), \gamma_1(t_1')]$ belongs to one of the *edges* of the square Q_1. We can reenumerate the other squares Q_s in such a way that I_1 would also belong to an edge of the square Q_2. In the second case (ii) the interval I_1 (except for its endpoints) lies strictly inside Q_1 (see Figure 2). In this case we do not reenumerate the other squares. Thus, in case (ii), for all $s \geq 2$ (and for all $s \geq 3$ in case (i)) we have $\gamma_1^{-1}(Q_s) \subseteq \gamma^{-1}(Q_s)$, and for all $s \geq 1$ we have $\operatorname{diam} \gamma_1^{-1}(Q_s) < \sqrt{3}$.

If Γ_1 intersects Q_2 at no more than one point (which is possible only in case (ii)), we set $\gamma_2 = \gamma_1$. Otherwise there exists a minimal closed arc T_2 with length less than $2\pi/3$ that contains $\gamma_1^{-1}(Q_2)$. Note that T_1 and T_2 either have disjoint interiors (case (ii)), or $T_1 \subseteq T_2$ (case (i)). In both cases there exists a homeomorphism $\theta \mapsto (\cos\theta, \sin\theta)$ of some interval $[\tau_2, \tau_2']$ onto T_2 and constants a_2, b_2, c_2, and d_2 such that the path $\gamma_2(t)$ that is equal to $\gamma_1(t)$ on $\mathbb{T} \setminus T_2$ and to $\gamma_2(t) = (a_2\theta + b_2, c_2\theta + d_2)$ for $t = (\cos\theta, \sin\theta) \in T_2$ (when $\theta \in [\tau_2, \tau_2']$) is a simple closed path. It coincides with γ at the starting point $t_2 = (\cos\tau_2, \sin\tau_2)$ and at the terminal point $t_2' = (\cos\tau_2', \sin\tau_2')$ of the arc T_2, because t_2 and t_2' do not lie inside T_1.

Continuing similarly, as a result we obtain simple closed paths γ_s, $s = 1, \ldots, S$. Suppose $t \in \mathbb{T}$; we estimate $|\gamma_S(t) - \gamma(t)|$. If $\gamma_S(t) \neq \gamma(t)$, then there exists $s \in \{1, \ldots, S\}$ such that $\gamma_S(t) = \gamma_s(t) \neq \gamma_{s-1}(t)$ (we put $\gamma_0 = \gamma$). By construction, t belongs to an arc T_s starting at t_s and ending at t_s', $\gamma_s(T_s) \subset Q_s$, $\gamma_s(t_s) = \gamma(t_s)$, $\gamma_s(t_s') = \gamma(t_s')$. Hence,

$$|\gamma_S(t) - \gamma(t)| = |\gamma_s(t) - \gamma_s(t_s) + \gamma(t_s) - \gamma(t)| \leq \delta + |\gamma(t) - \gamma(t_s)|\,.$$

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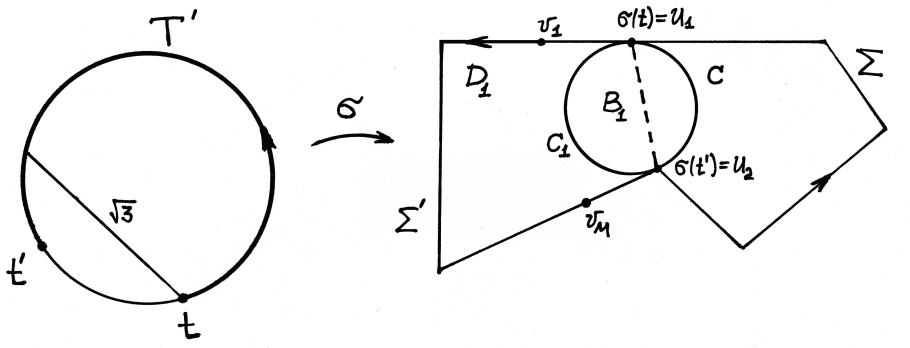

FIGURE 3. To the proof of Lemma 3.

But $|t - t_s| \le |t'_s - t_s| \le \varepsilon_1$ because $|\gamma(t'_s) - \gamma(t_s)| \le \delta \le \varepsilon_2$. Therefore,

$$\delta + |\gamma(t) - \gamma(t_s)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which implies finally that $|\gamma_S(t) - \gamma(t)| < \varepsilon$.

Since the intersection of $\Gamma_S = \gamma_S(\mathbb{T})$ with every square of the lattice is either empty, or consists of a single point, or consists of "uniformly" traversed line segment, it is easily seen that $\sigma = \gamma_S$ is the approximation we need. □

§3. Two main lemmas

Lemma 3. *Let Σ be a simple closed polygon with a realization (σ, Θ). Then in the bounded component D of the set $\mathbb{R}^2 \setminus \Sigma$ there exists an open disk B whose boundary C intersects the polygon Σ at points $\sigma(t)$ and $\sigma(t')$, where $t, t' \in \mathbb{T}$ satisfy $|t - t'| \ge \sqrt{3}$.*

Proof. We may assume that, when moving along Σ in accordance with the orientation of $\mathbb{T}$ and the map σ, the domain D remains on the left-hand side (otherwise we can use symmetry with respect to one of the coordinate axes). The Weierstrass theorem mentioned above allows us to find an open disk B in D whose boundary C intersects Σ at points $\sigma(t)$, $\sigma(t')$, where $t, t' \in \mathbb{T}$, such that $|t - t'|$ is maximal (see Figure 3; note that C can intersect Σ also at other points). We prove that this disk is the one we need. Suppose that $|t - t'| < \sqrt{3}$. Let T be an arc on $\mathbb{T}$ that joins t with t', has length greater than $4\pi/3$, and is oriented counter-clockwise (like $\mathbb{T}$). The point t will be the starting point, and t' the ending point of T. Obviously, the boundary C of B has no points in common with the part $\Sigma' = \sigma(T')$ of the polygonal line Σ, where $T' = T \setminus \{t, t'\}$. Without loss of generality we may assume that $v_1 = \sigma((\cos\theta_1, \sin\theta_1)), \dots, v_M = \sigma((\cos\theta_M, \sin\theta_M))$ are all (consecutive) vertices of Σ that belong to Σ', $1 \le M \le N$. Put $u_1 = \sigma(t)$, $u_2 = \sigma(t')$. It is clear (see Lemma 1) that $\Sigma' = \sigma(T')$ and segment $[u_1, u_2]$ (a chord of the polygonal line Σ) enclose some domain D_1. Put $B_1 = B \cap D_1$, $C_1 = C \cap D_1$.

First, suppose that the arc C_1 touches Σ' at both points u_1 and u_2 (i.e., the circle C touches the lines u_1v_1 and v_Mu_2).

In this case, at the touching point the vector $\overrightarrow{u_1v_1}$ (the vector $\overrightarrow{v_Mu_2}$, respectively) goes "out" of C ("into" C, respectively), leaving B on the left-hand side, while the line segments $[u_1, v_1]$, $[v_M, u_2]$ lie on the same side of the line u_1u_2.

Since C_1 does not intersect Σ', in the domain $D_1 \cup B \subset D$ we can find a disk B' that touches the polygonal line Σ at points u'_1 and u'_2 lying *inside* the segments $[u_1, v_1]$ and $[v_M, u_2]$, respectively. Moreover, the points u'_1 and u_1 (as well as u'_2 and u_2) can be

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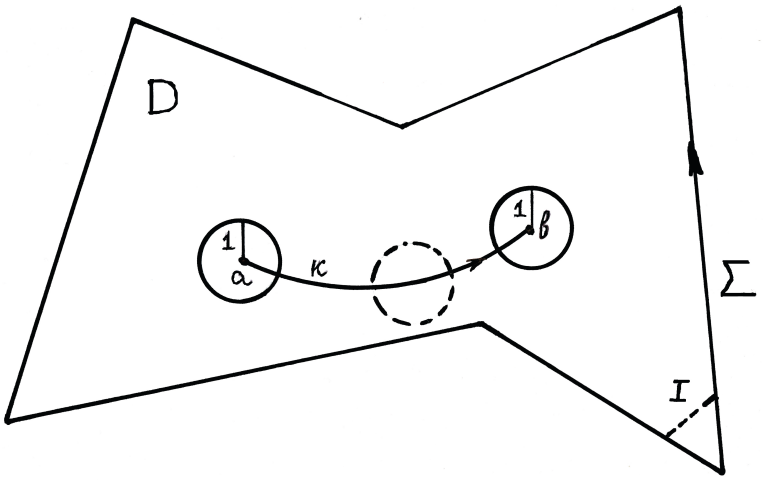

FIGURE 4

chosen to lie arbitrarily close to each other. But this contradicts the choice of B, because $|\sigma^{-1}(u_1') - \sigma^{-1}(u_2')| > |t - t'|$.

In the second case, we assume that C_1 touches the polygonal line Σ' at exactly one point, say, u_1 (the case of touching at the point u_2 is similar). In this case $u_2 = v_{M+1}$ is a vertex. Since the edge $[v_M, v_{M+1}]$ does not touch C_1, again we can choose a disk $B' \subset D_1 \cup B$ that touches Σ' at some point u_1' inside of $[u_1, v_1]$ (close to u_1), and whose boundary is close to u_2. We get a contradiction similar to that in the previous case.

Finally, suppose that both points $u_1 = v_N$ and $u_2 = v_{M+1}$ are vertices of Σ, and that C_1 does not touch Σ'. Let us "blow up" the disk B toward the domain D_1 in such a way that B still lie in D while its boundary C still contain the points u_1 and u_2. Then at some moment the arc C_1 will either touch the edge $[u_1, v_1]$, or touch the edge $[v_M, u_2]$, or intersect Σ'. We have already shown that each of these cases leads to a contradiction. □

Now we consider a simple closed polygon Σ and choose one of the components D of the set $\mathbb{R}^2 \setminus \Sigma$. For any chord I in D (i.e., any line segment between two distinct points on Σ that lies inside D except for its endpoints) the set $D \setminus I$ has two connected components (see Lemma 1). *We fix* points $a \in D$ and $b \in D$ such that $\mathrm{d}(\Sigma, \{a, b\}) \geq 1$. *Suppose* that for every chord I in D of length $\ell(I) < 2$ the points a and b belong to the same component of $D \setminus I$.

Lemma 4. *Under these assumptions, there exists a path (continuous map)* $\kappa\colon [0,1] \to D$ *between* a *and* b *(i.e.,* $\kappa(0) = a$ *and* $\kappa(1) = b$*) such that* $\mathrm{d}(\Sigma, K) \geq 1$*, where* $K = \kappa([0,1])$*; see Figure* 4.

Proof. Let A_a be the set of points in D that can be connected to a by paths κ_a (defined on $[0,1]$) such that $\mathrm{d}(\Sigma, K_a) \geq 1$, where $K_a = \kappa_a([0,1])$. Put

$$\Sigma_a = \{z \in \Sigma\colon \exists\, a_z \in A_a \ \text{ such that } \ |z - a_z| = 1\}.$$

For the point b the sets A_b and Σ_b are defined similarly. We shall prove that $A_a \cap A_b \neq \varnothing$ (which immediately implies that $A_a = A_b$). We may assume that when we move along Σ (in agreement with the orientation) the domain D is on the left-hand side.

We need the following lemma.

Lemma 5. *Under the above assumptions, we have* $\Sigma_a = \Sigma_b$*. Moreover,* Σ_a *has finitely many connected components (finitely many closed intervals and points on* Σ*).*

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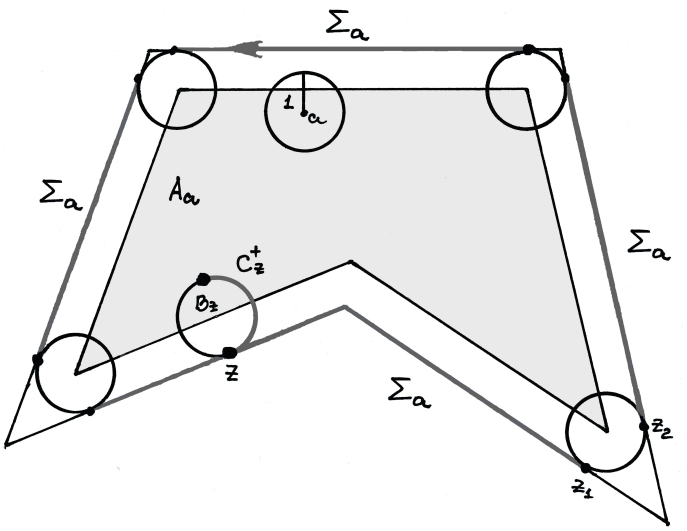

FIGURE 5. To the proof of Lemma 5.

Proof. Let $\bar{E}$ be the closure of a (nonempty) set $E \subset \mathbb{R}^2$. Suppose that $z \in \Sigma_a$, and let B_z be a unit circle with center at the point $a_z \in A_a$ and such that its boundary C_z contains z. Denote by C_z^+ the open half-circle on C_z with z as the starting point and going counter-clockwise.

First, suppose that z is not a vertex of the polygonal line Σ. Then B_z touches some edge of Σ that contains z. If $C_z^+ \cap \Sigma \neq \varnothing$, the point z (and the disk B_z with it) can be "moved" along Σ (in the direction determined by the orientation of Σ) until either z reaches a next vertex for the first time, or C_z^+ starts to intersect Σ. In the first case we proceed with continuous "rolling" of the disk B_z clockwise around the point z we reached until either the first moment when $C_z^+ \cap \Sigma \neq \varnothing$, or the first moment when C_z becomes tangent to the next edge after the vertex z. Proceeding like this, we shall reach the situation when $C_z^+ \cap \Sigma \neq \varnothing$ for the first time (generally speaking, the starting point $z \in \Sigma_a$ can occur somewhere in the middle of the described process). This last position of the point z (denote it by z_1) is the "extreme" point of the component of Σ_a that contains the starting position of z (see Figure 5).

Indeed, let z_2 be the closest (while moving from z_1 along Σ) point on Σ such that $z_2 \in C_{z_1}^+ \cap \Sigma$. We prove that the (open) interval Σ_{12}° of the polygon Σ starting at z_1 and ending at z_2 contains no points of Σ_a (see Figure 6). Furthermore, at the same time we shall prove that on Σ_{12}° there are no points of Σ_b (which implies that $\Sigma_b \subset \Sigma_a$ and, symmetrically, $\Sigma_a \subset \Sigma_b$, whence $\Sigma_a = \Sigma_b$), and Lemma 5 will be proved.

Suppose the contrary: there exist points $w \in \Sigma_{12}^\circ$ and $a_w \in A_a \cup A_b$ such that $|a_w - w| = 1$. Then the unit disk B_w (with boundary C_w and center a_w) lies entirely in D. Let $I = [z_1, z_2]$ be a chord in D. Since $|z_1 - z_2| < 2$, the set $A_a \cup A_b$ (and, consequently, the point a_w) lies in one component of D_1 bounded by $(\Sigma \setminus \Sigma_{12}^\circ) \cup [z_1, z_2]$. Since $a_w \in D_1$ and $w \notin \bar{D}_1$, the radius $[a_w, w)$ of the disk B_w must intersect the chord I, while not intersecting the set $\Sigma \setminus \Sigma_{12}^\circ$. Moreover, B_w does not contain z_1 and z_2, so that C_w intersects I at two points. Since a_w and a_{z_1} belong to A_a (where a_{z_1} is the center of the disk B_{z_1} whose semicircle $C_{z_1}^+$ contains z_2) and lie on the same side of $z_1 z_2$, we see that either $a_w = a_{z_1}$ (and the point $w \in C_{z_1}^+$ goes before z_2), or $w \in B_{z_1}$, which is impossible. □

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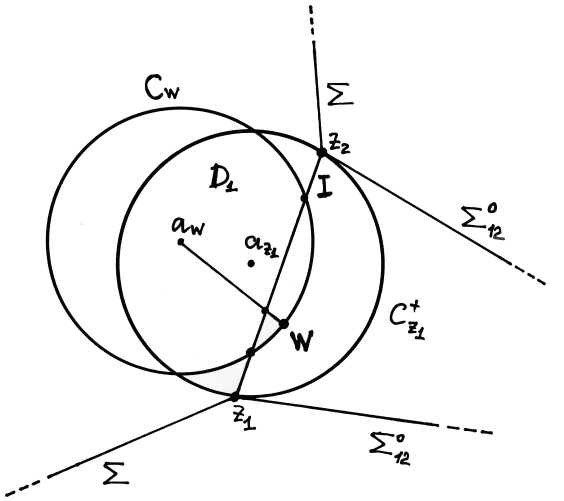

FIGURE 6. To the proof of Lemma 5, $w \in \Sigma^\circ_{12}$.

Now we can finish the proof of Lemma 4. Recall that $\Sigma_a = \Sigma_b$. We prove that $A_a = A_b$. Suppose that $z \in \Sigma_a = \Sigma_b$, $a_z \in A_a$, $b_z \in A_b$, and let B^a_z, B^b_z be the unit disks with centers at a_z, b_z, respectively, whose boundaries C^a_z, C^b_z contain the point z. The case where $a_z = b_z$ is clear. Otherwise the angle $\angle(a_z z b_z)$ is nonzero, so that the point z is a vertex of the polygonal line Σ. It follows that the disks B^a_z and B^b_z have nonempty intersection. Without loss of generality we may assume that the continuous rotation of the disk B^a_z around the point z that superposes B^a_z with B^b_z and goes "the shortest way" is clockwise. After this rotation, we either end up at B^b_z without intersecting B^a_z with the broken line Σ (then $b_z \in A_a$, and the proof is finished), or $z = z_1$ will be the extreme point of the component Σ_a containing z_1 (see Figure 7). In the latter case, suppose that z_2 is the point of Σ_a that follows after z_1 (as in Lemma 5), and that $a_* \in A_a$ is the center of the unit disk B_* whose boundary C_* contains the points z_1 and z_2, where $a_* \neq b_z$. Then B_* is the extreme position of the rotated disk B^a_z that does not intersect Σ. If the ray zb_z (with z as the initial point) lies between the rays za_* and z_1z_2, then the disk B^b_z contains z_2, a contradiction. On the other hand, if the ray z_1z_2 lies between the rays za_* and zb_z, then the points a_*, b_z lie in different components of $D \setminus [z_1, z_2]$, because the interval $[a_*, b_z]$, which lies in $B_* \cup B^b_z \subset D$, intersects the interval $[z_1, z_2]$ only once. Again we have a contradiction, which proves Lemma 4. □

§4. Proof of Jordan's Theorem

First, we prove that $\mathbb{R}^2 \setminus \Gamma$ has at least two components. It suffices to show the existence of a *bounded* component. Consider a sufficiently large disk B_0 that contains Γ, is centered at zero, and has boundary C_0. Let $\Gamma_1, \dots, \Gamma_m, \dots$ be simple closed polygons that converge to Γ (in the sense of Lemma 2), and let $\gamma_1, \dots, \gamma_m \dots$ be paths that converge to γ and realize $\Gamma_1, \dots, \Gamma_m \dots$, respectively. By Lemma 3, for every m there exists a disk B_m (with center b_m and boundary C_m) that lies in the domain D_m enclosed by the polygon Γ_m and possesses the property that there exist points $t_m \in \mathbb{T}$ and $t'_m \in \mathbb{T}$ with $|t_m - t'_m| \geq \sqrt{3}$ such that $\gamma_m(t_m) \in C_m$ and $\gamma_m(t'_m) \in C_m$. Passing to a subsequence if needed, we may also assume that all Γ_m's lie in B_0, and that the sequence $\{b_m\}$ converges to some point b as $m \to +\infty$.

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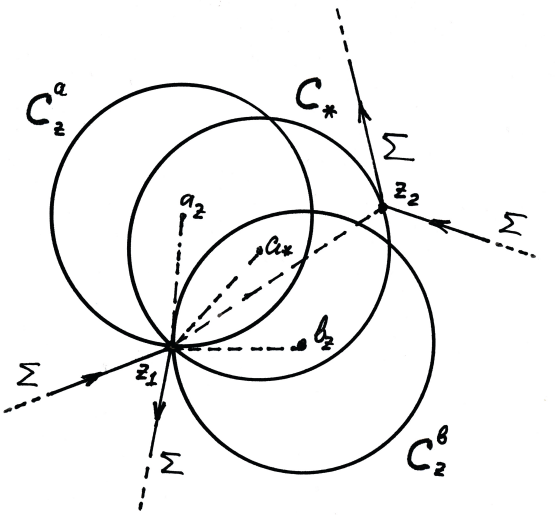

FIGURE 7. To the end of the proof of Lemma 4.

Choose $\varepsilon > 0$ such that the conditions $t,\ t' \in \mathbb{T}$ and $|t - t'| \geq \sqrt{3}$ imply that $|\gamma(t) - \gamma(t')| \geq \varepsilon$. Then $|\gamma(t_m) - \gamma(t'_m)| \geq \varepsilon$, whence $|\gamma_m(t_m) - \gamma_m(t'_m)| > \varepsilon/2$ for sufficiently large m. Therefore, $\operatorname{diam} B_m > \varepsilon/2$ and $\mathrm{d}(b_m, \Gamma_m) > \varepsilon/4$ for large m. It follows that, for large m, the points b_m and b lie in the same (bounded) component of the set $\mathbb{R}^2 \setminus \Gamma_m$ and, therefore, b_m and b lie in the same component of $\mathbb{R}^2 \setminus \Gamma$. If b_m and b lie in the unbounded component $\mathbb{R}^2 \setminus \Gamma$, then there exists a path $\kappa\colon [0,1] \to \mathbb{R}^2 \setminus \Gamma$ between b and C_0. Suppose that $\mathrm{d}(K, \Gamma) = \delta > 0$, where $K = \kappa([0,1])$. Since $|\gamma(t) - \gamma_m(t)| < \delta/2$ for all $t \in \mathbb{T}$ and all sufficiently large m, it follows that $\mathrm{d}(K, \Gamma_m) > \delta/2$, so that, for m sufficiently large, the points b_m and b lie in the unbounded component of $\mathbb{R}^2 \setminus \Gamma_m$, a contradiction.

Now we prove that $\mathbb{R}^2 \setminus \Gamma$ has at most two connected components. Suppose the contrary, let w_1, w_2, w_3 be three points in different components of $\mathbb{R}^2 \setminus \Gamma$. Suppose that $\mathrm{d}(\Gamma, \{w_1, w_2, w_3\}) = \varepsilon$ and that simple closed polygons $\Gamma_1, \dots, \Gamma_m, \dots$ converge to Γ, i.e., that $|\gamma(t) - \gamma_m(t)| \to 0$ as $m \to +\infty$ uniformly on $\mathbb{T}$. Then we may assume that $\mathrm{d}(\Gamma_m, \{w_1, w_2, w_3\}) \geq \varepsilon/2$ (for all m), so that two of the points w_1, w_2, w_3 lie in the same component of $\mathbb{R}^2 \setminus \Gamma_m$. We may assume that w_1 and w_2 lie in the same component D_m of the set $\mathbb{R}^2 \setminus \Gamma_m$ for all m. Suppose that there exist $\delta \in (0, \varepsilon)$ and infinitely many values of m such that w_1 and w_2 can be joined by a path $\kappa_m\colon [0,1] \to D_m$ such that $\mathrm{d}(K_m, \Gamma_m) \geq \delta$, where $K_m = \kappa_m([0,1])$. But then w_1 and w_2 must lie in the same component of $\mathbb{R}^2 \setminus \Gamma$. This contradicts our assumption, hence there is no such δ. Now we apply (the contrapositive of) Lemma 4. Passing to a subsequence once again, we may assume that for every m there exist points $z_m = \gamma_m(t_m)$, $t_m \in \mathbb{T}$, and $z'_m = \gamma_m(t'_m)$, $t'_m \in \mathbb{T}$, such that the points w_1 and w_2 lie in different components of the set $D_m \setminus I_m$, where $I_m = [z_m, z'_m]$, while $z_m - z'_m = \gamma_m(t_m) - \gamma_m(t'_m) \to 0$ as $m \to +\infty$. Therefore, $t_m - t'_m \to 0$ as $m \to +\infty$. Without loss of generality we may assume that for infinitely many values of m the point w_1 lies in the component D'_m of the set $D_m \setminus I_m$ that is bounded by I_m and $\gamma(T'_m)$, where T'_m is the minimal arc on $\mathbb{T}$ that joins t_m and t'_m. It is clear that $\operatorname{diam} D'_m \to 0$ as $m \to +\infty$, whence we see that the points w_1 must lie on Γ. This contradiction finishes the proof of Jordan's Theorem. □

The arguments and constructions we have used to prove Jordan's theorem allow us to derive a series of interesting and useful corollaries.

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Corollary 1. *Under the assumptions of Jordan's Theorem, put*

$$\delta := \min\{|\gamma(t) - \gamma(t')| \,:\, t, t' \in \mathbb{T},\ |t - t'| \geq \sqrt{3}\}.$$

Then the bounded component of the set $\mathbb{R}^2 \setminus \Gamma$ *contains a disk of diameter* δ.

Modifying Lemmas 1, 2, 4, and the second part of the proof of Jordan's Theorem in a natural way, we obtain the following important result.

Jordan's Theorem for a Jordan curve. *Let* $\gamma\colon [0,1] \to \mathbb{R}^2$ *be a continuous injective map, i.e.,* $\Gamma = \gamma([0,1])$ *is a Jordan curve. Then* $\mathbb{R}^2 \setminus \Gamma$ *is connected.*

Moreover, the following proposition holds.

Corollary 2. *Under the assumptions of Jordan's theorem, the boundary of each of the connected components of the set* $\mathbb{R}^2 \setminus \Gamma$ *is* Γ.

Proof. It suffices to consider the bounded component D of $\mathbb{R}^2 \setminus \Gamma$. Suppose, to the contrary, that the boundary ∂D of D does not coincide with Γ. Since, clearly, $\partial D \subset \Gamma$, it follows that for some $t_0 \in \mathbb{T}$ we have $\gamma(t_0) \notin \partial D$. Then there exists a connected neighborhood T_0 of the point t_0 in $\mathbb{T}$ such that $\partial D \cap \gamma(T_0) = \varnothing$. Meanwhile, the Jordan curve $\Gamma_1 = \gamma(T_1)$, where $T_1 = \mathbb{T} \setminus T_0$, contains ∂D and does not partition a plane (by Jordan's theorem for a Jordan curve). Now a contradiction can easily be derived from the Nested Intervals theorem. □

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